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STATISTICAL THEORY AND METHOD

VOLUME 2 · NUMBER 1

1961

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INTERNATIONAL JOURNAL OF ABSTRACTS
STATISTICAL THEORY AND METHOD

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THE aim of this journal of abstracts is to give complete coverage of papers in the field of statistical theory and new contributions to statistical method. Papers which report only applications or examples of existing statistical theory and method will not be included. There are approximately two hundred and fifty journals published in various parts of the world which are wholly or partly devoted to the field of statistical theory and method and which will be brought within the scope of this journal of abstracts. In due course it will be possible to issue a complete list of these journals. In the case of the following journals, however, being those which are wholly devoted to statistical theory—all contributions, whether a paper, note or miscellanea, will be abstracted :

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Biometrika
Journal, Royal Statistical Society (Series B)
Bulletin of Mathematical Statistics
Annals, Institute of Statistical Mathematics

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Biometrics
Metrika
Metron
Review, International Statistical Institute
Technometrics
Sankhyā

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In addition to the ordinary journals, there are two kinds of publication which fall within the scope of this journal of abstracts. They are the experiment and other research station reports—which occur particularly in the North American region—and the reports of conferences, symposia and seminars. Whilst these latter may be included in the book review sections of journals it is unusual for any individual contribution to be noted at any length. These publications are, in effect, special collections of papers and for this reason the appropriate arrangements will be made for them to be included in this journal. By the same token, abstracts of papers given at conferences and reproduced in an appropriate journal will be disregarded until the definitive publication is available.

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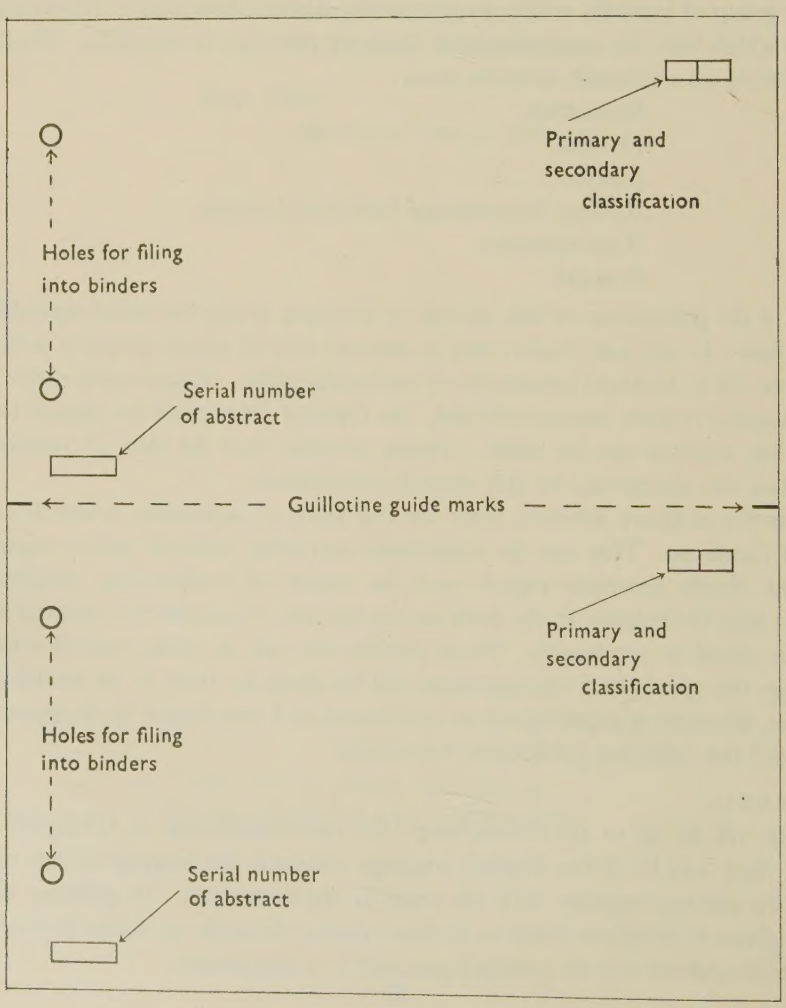
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A Journal of The Royal Statistical Society

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2	**0.6**	Akaike	Journal page reference should read 11-20.
199	**3.9**	Mickey	Secondary classification should read (5.0).
393	**3.0**	Zia-Ud-Din	Journal page reference in second line should read 253-260.
417	**5.5**	Siotani	Secondary classification should read (5.8).
435	**6.2**	Lancaster	Symbols in line 6 should read $N(\boldsymbol{O}, \boldsymbol{V})$.
440	**6.9**	Pawlik	Second displayed expression should read $0 \leqq \phi^2/(q-1) \leqq 1$.
			The character Λ in the first and third displayed expressions, and at end of line 7, should read 1.
496	**10.5**	Hatori	Line 12 should read . . . where λ is a positive constant.
531	**11.8**	Tedeschi	Abstractor's name is A. Del Chiaro.
543	**1.5**	Erdös & Rényi	Col. 2, line 2 should read $D_N(x)$.
556	**1.0**	Rényi	Line 17, displayed expression should have semi-colon in place of equality sign after first (summation) term.
			Line 19, L.H.S. of displayed expression should read $\operatorname{Lim}_{x \to 1-0} \sum_{k=0}^{\infty} a_k \{x^{2^k}/(1-x^{2^k})\}$
557	**1.5**	Rényi	Line 18, displayed expression should read $\sigma_n = \sum_{k=1}^{n} \tau_k / B_n$ and, in lines 19 and 20, for τ_n read σ_n.
567	**2.4**	Bahadur	Centre symbol of displayed expression should be in italic: p^r.
580	**2.9**	Rényi	Line 5, R.H.S. of displayed expression should read $\frac{1}{2} n^{n-1} \sum_{k=3}^{n} \binom{n}{k} (k!/n^k)$
587	**3.8**	Rogers	Line 8, first term in displayed expression should read x_1.
598	**4.3**	Irwin	Line 13, the summation in the displayed expression should read $\sum_{r=1}^{\infty} \{r^{r-1}(\bar{x}e^{-\bar{x}})^r/r!\}$
623	**5.3**	James	Name in title of paper and in line 5 of abstract should read Behrens-Fisher.
628	**5.0**	Pfanzagl	Page numbers in reference given in line 5 from end should read 417-451.
658	**6.0**	Rényi	Line 3 from end, R.H.S. of first expression should commence with S.
Author Index No. 1 & 2			White, R. F.: change intitials to J. S.
Author Index No. 3			Pawlik, E.: change initial to K.
Author Index No. 4			Milicer-Gruzewska, A.: change initial to H.
			Tucker, M. G.: change initials to H. G.

BOYD, A. V. (University of the Witwatersrand, South Africa) **0.4** (–.–)

Inequalities for Mill's ratio—*In English*

Rep. Statist. Appl. Res. (JUSE) (1959) **6,** 44-46

Let $R(x)$ denote Mill's ratio, $e^{\frac{1}{2}x^2}\int_x^\infty e^{-t^2/2}\,dt$ for $x \geqq 0$. In this paper, the form $\alpha[(x^2+\beta)^{\frac{1}{2}}+\gamma x]^{-1}$ is considered in an attempt to find a good approximation to $R(x)$. The author has shown that for the approximation to be good for both large and small x, it is necessary that $\alpha = \gamma+1$ and $\alpha\beta^{-\frac{1}{2}} = (\pi/2)^{\frac{1}{2}}$, that is, the functional form to be considered is

$$\Pr(x, \gamma) = (\gamma+1)[\{x^2+2(\gamma+1)^2/\pi\}^{\frac{1}{2}}+\gamma x]^{-1}.$$

For this functional form, the author has proposed $\Pr[(x, 2/(\pi-2)]$ as the upper bound and $\Pr(x, \pi-1)$ as the lower bound to $R(x)$.

(M. Siotani)

2/1

DANZIG, G. B. (RAND Corp., Santa Monica, Cal.) **0.8** (–.–)

On the status of multi-stage linear programming problems—*In English*

Bull. Int. Statist. Inst. (1958) **36,** III 303-320 (30 references)

This paper reviews work done in its chosen field during the period 1949 to 1956 with the earlier paper by Hitchcock [*J. Math. Phys.* (1941) **20**, 224-230]. After an introduction which poses the multi-stage dynamic situation in terms of the well-known warehousing problem, the author considers the functional equation approach. In this he elaborates a proposal made by Bellman [RAND Corporation paper p. 918, (1956)]. Concluding that, when the stages of a multi-stage situation are tied together by more than one variable, the factorial equation approach becomes increasingly tedious to apply—the author next considers dynamic Leontieff models with substitution.

The next part of this paper states that " one of the most exciting ideas to date is a possibility that the solution to a dynamic problem might be obtained as a by-product of an iterative procedure for solving a steady state problem ". " The interesting open question is whether this idea can be generalised ". The author then considers the need to solve large scale systems [see also *Management Sci.* (1956) **2**, 131-144] and noted that, for linear programming problems involving matrices which exhibit a special structure, it seems possible to develop special techniques that could extend many times the size of systems capable of being solved by general basic programming techniques. An example is given relating to the problem of routing cargo aircraft: programming an industrial complex—perhaps automobile production —is another example.

The final section of the paper deals with the problem of solving general block-triangle systems: the majority of practical problems fall into the block-triangle class. A number of variants of the simplex method have been suggested to cut the number of iterations: references are given to the work of Beale; Orchard-Hays; Danzig; Ford; Fulkerson; and Markowitz. The paper concludes by reviewing proposals by Markowitz and by the present author for finding a compact form for the inverse of the basis.

(W. R. Buckland)

2/2

DANZIG, G. B. (Report of discussion on a paper by) **0.8** (–.–)
On the state of multi-stage linear programming problems—*In English*
Bull. Int. Statist. Inst. (1959) **36,** I 122-124,

Those taking part in the discussion were Hirsch, de Wolff, and Frisch: the author replied briefly.

Hirsch drew attention to some problems with regard to the computational theory of linear programming which had not yet received sufficient attention from mathematicians. For example the problem of determining *a priori* non-trivial upper bounds for the number of iterations required by the scheme was mathematically formidable. The simple method which had been in use for ten years still did not have a serviceable upper bound for the number of iterations involved. With regard to the paper, the author had not discussed the approach of dynamic programming using parametric linear programming. It should be noted that the technique of transforming the dynamic warehousing problems into a static network flow problem was not generally applicable to multi-stage problems when special techniques might be employed in a powerful way.

de Wolff raised the point that, in working with a large number of equations and variables, the data had to be available to a greater number of decimal places than justifies by its intrinsic accuracy.

Frisch commented upon the point raised by de Wolff using some concepts connected with the multiplex method as developed in the Oslo University Institute of Economics.

(W. R. Buckland)

2/3

GREENBERG, B. G. & SARHAN, A. E. (Biostatistics Dept., University of North Carolina) **0.6** (–.–)
Matrix inversion, its interest and application in analysis of data—*In English*
Bull. Int. Statist. Inst. (1960) **37,** III 107-120 (7 references)

Matrix inversion is used in the least-squares analysis of data to estimate parameters and their variances and covariances. When the data come from the analysis of variance, analysis of covariance, order statistics, or the fitting of response-surfaces, the matrix to be inverted usually falls into a structured pattern that simplifies its inversion.

One matrix pattern that occurs frequently when estimating parameters in order statistics is an arrangement whereby the element in each row, from the main diagonal on the right, is a constant multiple of the corresponding element in that column in the first row. That is, $v_{ij} = \lambda_i v_{1j}$ for $i \leqq j$. When this occurs, the inverse is a diagonal matrix of type 2, that is $v_{ij}^{-1} = 0$ for $|i-j| \geqq 2$. The inverse solution is obtainable, furthermore, in terms of λ_i and v_{ij}.

Another arrangement that extends this concept is that in which the elements of each row, starting with the third, are functions of the first two rows. That is, $\theta_{1i} v_{1j} + \theta_{2i} v_{2i} = v_{ij}$. The inverse of this matrix pattern is a diagonal matrix of type 3, or $v_{ij}^{-1} = 0$ for $|i-j| \geqq 3$. A general solution is also available, and these results are applied to a problem in order statistics using the two-parameter exponential distribution.

Patterned matrices also are amenable to partitioning and this is another convenient device to find the exact inverse quickly. In fitting a response-surface to data, for example, a complicated looking matrix can be abbreviated and simplified by selective partitioning. When this has been done, the exact inverse can be found by equating the product of the matrix and its inverse to the elements of the identity matrix. This procedure is illustrated with some data from a problem in the fitting of a response surface.

When the matrix has no special pattern, as in the usual regression problem, the recommended procedure for matrix inversion is that of the modified square-root method.

(B. G. Greenberg)

2/4

OTTAVIANI, G. (Statistical Faculty, University of Rome) **0.1** (–.–)

Concerning the solution of an integral equation due to Cantelli, relating to Mathematical Statistics—*In Italian*

G. Ist. Ital. Attuari (1959) **22,** 11-39 (7 references)

In this note the author considers a problem which was posed in 1917 by Cantelli, and is concerned with the research into possible systems (X, Y) of variates for which

(*a*) the variate X is Gaussian;

(*b*) if the variate X takes the value x, Y is a Gaussian variate with a mean value equal to zero, and root-mean-square deviation given by $\sqrt{f(x)}$, $f(x)$ being an assigned positive function definite within the whole interval $(-\infty, +\infty)$ of the real variable x;

(*c*) the variable $X+Y$ is Gaussian.

If $\sqrt{\mu}$ denotes the root-mean-square deviation of X and if

$$f=\int_{-\infty}^{+\infty}\frac{e^{-\frac{x^2}{2\mu}}}{\sqrt{2\pi\mu}}f(x)dx>0,$$

the problem becomes an integral equation obtained by equalising the density of the probability of the variates $X+Y$ to the density of the probability of a Gaussian variate Z, with a mean value equal to zero and root-mean-square deviation $\sqrt{\mu+f}$.

An obvious and particular solution is given by $f(x) = a$ constant, which corresponds with the case of two variates X, Y independent of each other and Gaussian, which notoriously have a Gaussian variable as their sum. The author demonstrates that in the field of the functions $f(x)$ expandable into series of powers of x, convergent on the whole real x axis, and which take the absolute minimum value only in correspondence with finite values of x, no other solution exists but the one referred to with two independent aleatory variables. It is then deduced that if the function $f(x)$ is expandable into series of powers convergent on the whole x axis, and if it admits a limit both for $x\to-\infty$ and for $x\to+\infty$, then it cannot be a solution of the problem, unless it is constant.

Finally, the author observes that the demonstration given is perhaps adaptable also when $f(x)$ has not its minimum value in the finite and with only the functions $f(x)$ measurable and that in any case one can tell intuitively that the only solution is the constant. Attention is called, however, to the fact that such functions are to be rejected *a priori* in the work of research; in research work the problem under consideration is a static problem suggested by real phenomena.

2/5 (A. Del Chiaro)

SAKAGUCHI, M. (University of Electro-Communications, Tokyo) **0.7** (0.8)

On a certain multi-stage game—*In English*

Rep. Statist. Appl. Res. (JUSE) (1959) **6,** 1-4 (1 reference)

The author treats a multi-stage game of some simple structure and gives an optimal strategy in the sense of the minimax principle using the dynamic programming technique. The multi-stage game considered here is as follows. Let $\{E_i\}(i = 1, 2, \ldots, n)$ be a system of random events which are mutually exclusive and let the probability of occurrence of each event E_i be p_i and $\sum_{i=1}^{n} p_i = 1$. Under the condition that the probabilities $\{p_i\}(i = 1, \ldots, n)$ are unknown to a gambler, he is required to place bets on the events. He is to bet a quantity of money z_i on the ith event E_i subject to the restrictions that $\Sigma z_i = x$, his initial capital, and, if E_i occurs, he receives $r_i z_i$, otherwise nothing. This process is continued for N stages, with a pay-off at the end of each stage. The gambler wishes to maximise the expected value of a utility function $\phi(w)$ of the final total at the end of the process.

For this multi-stage game, the author obtains the maximum-minimum policy in the case where $\phi(w)$ is continuous, non-decreasing and concave. According to the policy, betting at each stage is independent of the number of subsequent stages and of the amount of money available; that is, the policy is to bet $\left(\frac{1}{r_i}\Big/\sum_{i=1}^{n}\frac{1}{r_i}\right)$ times the amount of money available on $E_i(i = 1, 2, \ldots, n)$.

(Y. Suzuki)

SAKAGUCHI, M. (University of Electro-Communication, Tokyo)

Values of strategic information—*In English*

Rep. Statist. Appl. Res. (JUSE) (1959) **6,** 5-12 (4 references, 5 figures)

In this paper, the author defines values of strategic informations concerning various information patterns of a matrix game and proves a " higher order " min-max theorem in a linear programming approach. Let two players be P_1 and P_2 and their pure-strategy spaces $S_1 = \{\alpha_1, \ldots, \alpha_m\}$ and $S_2 = \{\beta_1, \ldots, \beta_m\}$, respectively. Further, let the payoff matrix be $A = (a_{ij}/i = 1, \ldots, m; j = 1, \ldots, n)$.

Then, consider the decompositions $\mathrm{P}_1 = \{S_{1k}\}$ and $\mathrm{P}_2 = \{S_{2l}\}$ of S_1 and S_2 respectively, and denote the sub-matrices of A corresponding to $S_1 \times S_{2l}$ and $S_{1k} \times S_2$ by A^l and A_k. Further, define $\max_x \min_y xAy = v(A)$ where x and y are mixed strategies of P_1 and P_2, respectively. Under the situation where P_1 moves first and an umpire tells P_1 at his move the set S_{2l} to which the pure strategy chosen by P_2 belongs, if P_2 is a rational player, then the average gain of P_1 is $\min_l v(A_l)$. On the other hand, if there does not exist such an umpire and each player cannot get any information about the opponent player's strategy and they are rational, the average gain of P_1 is $v(A)$. The difference $J_1(\mathrm{P}_2) \equiv \min_l v(A^l) - v(A)$ is defined as a measure of the amount of the information given by the umpire to P_1.

In Theorem 1, the author shows that $J_1(\mathrm{P}_2)$ is non-negative and not larger than $\min_{1 \leqq j \leqq m} \max_{1 \leqq i \leqq m} a_{ij} - v(A)$. In particular if the decomposition P_2 is the best one, that is, $\mathrm{P}_2 = \{\beta_1, \beta_2, \ldots, \beta_n\}$, $J_1(\mathrm{P}_2)$ attains its maximum value and, on the contrary, if $\mathrm{P}_2 = \{S_2\}$, then $J_1(\mathrm{P}_2) = 0$.

In the subsequent part of the paper, the author treats the situation where there is no umpire as is mentioned above, but the player P_1 knows the habit of choice of strategies by the opponent player P_2. The habit of choice is defined using decomposition $\mathrm{P}_2 = \{S_{2l} \mid (l = 1, \ldots, v)\}$ and probability vector $q = (q_1, \ldots, q_v)$ in such a way that P_2 has a tendency to choose his strategy from S_{2l} with probability q_l. The set of all strategies produced according to this habit of choice is denoted by $\mathrm{Y}_{\mathrm{P}_2, q}$. Define $X[A]$ and $Y[A]$ as the sets of all max-min and min-max strategies of the matrix game A, respectively. Then, the value of information obtained by knowing the

SAKAGUCHI, M. (University of Electro-Communication, Tokyo)

Values of strategic information—*In English*

Rep. Statist. Appl. Res. (JUSE) (1959) **6,** 5-12 (4 references, 5 figures)

habit of choice of P_2 is defined as

$$I_1(\mathrm{P}_2, q) = \max_x \min_{y \in Y_{\mathrm{P}_2, q}} xAy - \min_{y \in \mathrm{P}_2, q} \max_{x \in X[A]} xAy (\geqq 0).$$

Further, if P_1 knows the habit of choice of P_2 and P_2 moves first and then an umpire tells P_1 at his move the set S_{2l} to which the choice made by P_2 belongs, the value of information is defined as

$$J_1(\mathrm{P}_2, q) \equiv \sum_{l=1}^{v} q_l v(A^l) - \min_{y \in Y_{\mathrm{P}_2, q}} \max_{x \in X[A]} xAy.$$

In Theorem 2, the author proves that $I_1(\mathrm{P}_2, q) \leqq J_1(\mathrm{P}_2, q)$.

In the final section, the author proves a " higher order " min-max theorem, that is,

$$\max_{x \in X_{\mathrm{P}_1, p}} \min_{y \in Y_{\mathrm{P}_2, q}} xAy = \min_{y \in Y_{\mathrm{P}_2, q}} \max_{x \in Y_{\mathrm{P}_1, p}} xAy,$$

using a linear programming technique, where $X_{\mathrm{P}_1, p}$ is an analogous one of P_1 to $Y_{\mathrm{P}_2, p}$ of P_2. This technique also shows us how to obtain a saddle point of the game.

(Y. Suzuki)

SINGH, D. (Indian Agricultural Research Inst., New Delhi) **0.2** (11.1)

Tables of orthogonal polynomials when the independent variable " x " is in the geometric progression: $x_r = 2^{r-2}$; $x_1 = 0$; $x_r = 2^{r-1}$; $x_1 = 1$—*In English*

J. Indian Soc. Agric. Statist. (1958) **10,** 131-140 (2 references, 4 tables)

Tables for fitting orthogonal polynomials to data which are equi-spaced have been given by Fisher & Yates [*Statistical Tables for Biological, Agricultural and Medical Research* (1943) Oliver & Boyd] and Aitken [*Proc. Roy. Soc. Edinb.* (1933) **53**, 54]. In certain biological and agronomic experiments the independent variable is found to be in a geometric series. The present tables have been worked out to meet this need. The geometric series with the common ratio 2 which is most widely used in biological research is considered in the present paper.

Detailed tables are worked out in two cases. In the first, the independent variable x is of the form $x_r = 2^{r-2}$, $x_1 = 0$ and in the second, $x_r = 2^{r-1}$, $x_1 = 1$.

If y is the dependent variable, we have to fit a polynomial of the form $y = b_0Z_0 + b_1Z_1 + b_2Z_2 + \ldots$ where Z_p is polynomial of the pth degree in x and $\Sigma Z_pZ_q = 0$, $p \neq q$ and $Z_0 = 1$. By the method of least-squares the solution for Z_p can be obtained as

$$\left(\sum_{s=0}^{p} d_{p,\,p-s} x^{p-s} \right) / d_{p,\,p}.$$

The d's are tabulated for $n = 3$ to 8 (n being the number of observations) up to the fourth degree for both the cases mentioned earlier.

Further, the solution for b can be expressed as $b = Lb'$ where $b'_2 \; \Sigma yz'/\Sigma z'^2$ and $L = z'/z$. The values of z' and L are tabulated for $n = 3$ to 8, for both the cases. For $n = 3$, 4, 5 these constants are tabulated up to the second, third and fourth degrees respectively while for $n = 6$, 7, 8 they are tabulated up to the third degree only.

(Y. R. Rau)

2/9

TALACKO, J. (Marquette University, Milwaukee, U.S.A.) **0.8** (–.–)

On stochastic linear inequalities—*In English*

Trab. Estadíst. (1959) **10,** 89-111 (57 references, 8 tables, 1 figure)

This review paper is concerned with some actual problems on the theory and methods of the linear programming; more explicitly with the statistical theory of the linear programming. Let $\boldsymbol{A} = (\alpha_{ij})$, be a rectangular matrix of order m, n; $\{\beta_i\}$ a column vector of order m; $\{\gamma_j\}$ a row vector of order n and $\sigma \leqq \boldsymbol{X} = \{x_1 \ldots x_n\}$, where $\boldsymbol{X}$ is a column vector. The problem of linear programming consists in finding one vector $\boldsymbol{X}$ for which the relations $\boldsymbol{AX} \leqq \beta$; $F(X^*) = \text{Max } \gamma \boldsymbol{X}$ hold.

If all the elements of $\boldsymbol{A}$, β and γ are constants (parameters) we have a deterministic model with a unique value for $F_0(X^*)$: X^* being in general a finite set. In several cases this situation is not real, because the elements of $\boldsymbol{A}$, β and γ can be random variables. The simplest case is that of a multivariate normal distribution; $\boldsymbol{A}$ is considered to be a sample of a normal population. For this case some bibliographic references are given.

The theory reviewed in the present paper is essentially non-parametric. Firstly A and γ are supposed to be constants and β variable; afterwards the theory is generalised to the case in which the three are variables. The solution is given by finding an upper bound for the probability that the optimum value of the linear programming is contained in a determined interval. The solution is found for the case in which the parameters have a Poisson distribution by means of confidence intervals.

Finally, two cases with some interesting results are given:

(i) A deterministic case with estimation by one point.

(ii) A stochastic case in which the greatest part of the elements varies; the general case in which $\boldsymbol{A}$, b and c are variables is first considered; secondly, and in order to compare the results, the cases in which a_{ij} varies and b and c are constants; b_i varies and a_{ij}, c_j are constants; c_j varies, a_{ij}, b_i are constants.

(J. Bejar)

2/10

TORNQVIST, L. (University of Helsinki) **0.8** (0.1)

Some new principles for solving linear programming problems—*In English*

Bull. Int. Statist. Inst. (1958) **36**, III 335-356 (4 references, 11 tables)

Using the theory of sets, the author deals formally and exhaustively with the problem of finding non-negative solutions to optimisation formulae under various assumptions. He derives exact criteria for the elements $h \in \hat{H}$ and $k \in \hat{K}$ corresponding to the optimal values $\hat{x}_k > 0$ and optimal multipliers $\hat{\lambda}_h > 0$, and gives substitute rules for use when the exact ones are difficult to apply.

A detailed examination is made in Appendix I of the familiar transportation problem and there are a number of numerical examples of this and other problems. A further appendix (II) sets out the multiplier technique used in the transportation problem. Some remarks about the probability of the existence of finite optimal solutions to linear programming problems, and about the distribution of $(\hat{x}, \hat{\lambda})$, are made in Appendix III.

(D. J. Page)

TORNQVIST, L. (Report of discussion on a paper by) **0.8** (0.1)

Some new principles for solving linear programming problems—*In English*

Bull. Int. Statist. Inst. (1959) **36**, I 124-126

The contributors to this discussion were Vajda and Castellano: the author replied briefly.

Vajda drew attention to the fact that the general approach by the author was towards speeding-up operations which looked rather slow by the Simplex Method—even though it could be argued the merit of the Simplex Method was that it proceeded by a number of easy steps rather than by a few complex ones. The author listed criteria for establishing a set of final positive variables. If this was completely achieved then the final solution was of a set of linear algebraic equations. However, it was possible that in practice the appropriate criteria might be difficult to identify in order that they might be applied.

Castellano drew attention to a paper given at a statistical meeting in Rome (May, 1951) when Herzel showed that the values which satisfied the conditions for the minimising of a fraction were given by the co-graduation table of Salvemini.

(W. R. Buckland)

VAJDA, S. (Admiralty Research Labs., Teddington, England) **0.8** (-.-)

Inequalities in stochastic linear programming—*In English*

Bull. Int. Statist. Inst. (1958) **36,** III 357-363 (8 references, 2 figures)

This paper is concerned with linear programming where the coefficients are subject to probability distributions. Two types of problem are discussed: firstly, the coefficients of the objective function have a probability distribution; secondly, the right-hand side set of the constraints are variable, and a linear term is incorporated in the objective formation.

In the first section, the author deals with the first of the above problems. He shows that the expected value of the minimum objective function is greater than or equal to the minimum objective functions with constant coefficients.

In the second section the author considers the second type of problem. He shows that with an unrestricted variable added to the constraints and with variable coefficients in the objective functions, the minimum objective function is less than or equal to the solution of the same problem with no variables.

In the third section, the variables added to the constants are constrained to be greater than zero. This can be thought of as the case where a fine must be paid if the constraints are not precisely met, but they must on no account be exceeded. The solution of this problem shows that the minimum objective function depends on the weighting function of the constant variables as they appear in the objective function.

The author gives clear numerical examples in the last two sections of the paper.

(A. N. James)

2/13

YOKOTA, T. (Institute of Statistical Mathematics, Tokyo) **0.1** (10.1)

Stochastic methods of solving partial integro-differential equations and their application to non-stationary Markoff process: I—*In English*

Ann. Inst. Statist. Math., Tokyo (1959) **10,** 137-161 (8 references)

The operator calculus of ordered exponentials is discussed; these are called the formal solutions of a partial integro-differential equation $\partial\Psi(x, s)/\partial s = A(s)\Psi(x, s)$, where the operator $A(s)$ is not always Hermitian. In the case of the Schroedinger equation appearing in quantum mechanics, the A is Hermitian operator and the ordered exponentials $\exp_{\pm} i \int_{t_0}^{t} A(s)ds$ are unitary operators. In the case of the Fokker-Planck equation appearing in stochastic theory, the A is usually non-Hermitian. The calculus described in this paper is applicable to the both cases. In the case of non-stationary Markoff process, $A(s)$ is not commutable with $A(s')$ for $s \neq s'$ and the calculus of ordered exponentials is different from that of commutable quantities. In this paper, fundamental formulae of calculation of ordered exponentials are derived and the foundations of the path integral method which is applicable to non-stationary Markoff processes are also given.

(T. Yokota)

2/14

BRAUMANN, P. (Faculty of Sciences, Lisbon) **1.1** (–.–)

Comments on the representation of infinitely divisible probability laws—*In German*

Rev. Fac. Cienc. Lisboa A (1959) **7,** 347-357

This paper is closely connected with another paper by the same author published in the same issue of the journal concerned (pages 255-262) under the title of " Symmetrische unbeschraenkt teilbare wahrscheinlichkeitgesetze " (abstracted in this journal No. 349, **1.1**). Some examples and connections are given relating the different representations of infinitely divisible laws and of their characteristic functions.

(J. T. de Oliveira)

GEIRINGER, Hilda (Harvard University, Mass., U.S.A.) **1.5** (2.5)

On a limit theorem to a compound Poisson distribution—*In English*

Math. Zeit. (1960) **72,** 229-234 (2 references)

In a former publication [*Zeit. angew. Math. Mech.* (1928) **8**, 292-309] the author had generalised the Poisson limit theorem to sums of independent random variables with discrete distributions over the values 0, 1, 2, ...m; m is arbitrary but fixed, where the probabilities for each of the values 1, 2, ...m are for $n\to\infty$ in the order of $1/n$. For the limit distribution of these sums the author now derives the probability generating function which is that of a compound Poisson distribution. Thus the generalisation of the simple Poisson limit does not lead again to a Poisson distribution—as in the case of the central limit theorem—but to a compound Poisson distribution. This is a remarkable result which can be extended to general discrete distributions taking on also negative values.

The problems can be illustrated by the following urn-scheme: Given n urns containing balls with the numbers 0, 1, 2, ...m in random order. Out of every urn one ball is taken and the numbers are summed up over all the n balls. The problem is to find the distribution of the sum, when n tends to infinity and the proportions of the balls with number 1, 2, ...m vanish as $1/n$. In the former publication the author had proved, that the limit distribution can be obtained by another urn-scheme, where nm urns are given, n of which contain only balls with the numbers 0 and 1, n contain only balls with the numbers 0 and 2, etc. ... and n contain only balls with the numbers 0 and m. Out of every urn one ball is taken at random and the numbers of these balls are summed up. In the limit, as $n\to\infty$ the proportions of all numbers except 0 again vanish as $1/n$.

This latter scheme is a model for the compound Poisson distribution. The new result is in accordance with the former results and extends them to the case of negative values.

(B. Schneider)

JOSEPH, A. W. & BIZLEY, M. T. L. (Wesleyan & Gen. Ass. Soc., and Bacon & Woodrow, Birmingham and London) **1.3** (0.4)

The two pack matching problem—*In English*
J. R. Statist. Soc. B (1960) **22**, 114-130 (5 references, 7 tables)

The authors envisage two packs of N cards, each pack containing cards of n different sorts, each having a_i of the ith sort, $i = 1, \ldots, n$ so that each is characterised by the vector $\boldsymbol{a} = (a_1, \ldots, a_n)$. One pack is arrayed in fixed order and all the $N!/\Pi a_i!$ ways of allocating the cards of the other, one to each card of the first, are considered and the exact valuation of the number which is denoted $\{r, \boldsymbol{a}\}$ of arrangements giving just r out of N pairs wherein both cards are of the same sort is investigated and in particular $\{0, \boldsymbol{a}\}$ which is denoted $\{\boldsymbol{a}\}$. Firstly the Kaplansky-Riordan expression for $\{\boldsymbol{a}\}$ as the result of a product of Laguerre polynomials in E, the increment operator, acting on $x!$ and then putting $x = 0$ is proved inductively and a new form of generating function for $\{\boldsymbol{a}\}$ is obtained from this. One of Macmahon's generating functions is compared and an expression for $\{\boldsymbol{a}\}$ as a single sum is hence derived in the case $n = 3$. Next $\{a_1, \ldots, a_{n+1}\}$ is expressed recursively as a weighted sum (over $r = 0, 1, \ldots, a_{n+1}$) of $\{r, a_1, \ldots, a_n\}$ by a combinatorial argument and this is used to find an expression for the generating function of $\{r;\ \boldsymbol{a}\}$ (*qua* function of r) from which an expression of Greville's for $\{r; \boldsymbol{a}\}$ may be deduced.

A list of some useful formulae relating to the action of Laguerre polynomials in E, acting on $x!$ at $x = 0$, is given with an outline proof of the more difficult ones and these are used to obtain some generalisations of Macmahon's recurrences for $\{\boldsymbol{a}\}$ which are for fixed n, in terms of increments in a_1, a_2 only: and, a combinatorial proof is given for some of these. Special formulae when all (or all but one) of the a_i are equal to a common value of 2, 3, 4 or 5 are derived. Tables of $\{\boldsymbol{a}\}$ are given for $n = 3, 4, 5$ and a to weight 5_n, largest part $\leqq 5$, with special tables of $\{a^n\}$, for a up to 12, 9, 7, 6 respectively when $n = 2, 3, 4, 5$. It is noted that whilst it is practicable to compute $\{r;\ \boldsymbol{a}\}$ from the tables of $\{\boldsymbol{a}\}$ there is an " arithmetically superior " method of finding $\{r;\ a_1, \ldots, a_{n+1}\}$ from $\{r; a_1, \ldots, a_n\}$ recursively, which is described and illustrated.

2/17 (D. E. Barton)

MILES, R. E. (Statistical Laboratory, Cambridge) **1.0** (1.3)

The complete amalgamation into blocks, by weighted means, of a finite set of real numbers—*In English*
Biometrika (1959) **46**, 317-327 (5 references, 1 table)

This paper is concerned with the problem of the " amalgamation " of a set of n real numbers $X_1, \ldots, X_n$ with an associated set of positive weights $W_1, \ldots, W_n$. This is a process wherein two successive X's which are in ascending order (of size) are replaced by their weighted mean for example $(X_1W_1 + X_2W_2)/(W_1 + W_2)$ and the corresponding weights by their sum; the process being repeated until the resultant sequence consists of k values, all in *descending* order. Thus k is some number between 1 and n. It is first shown, by elementary methods, that whatever the choice of successive pairs, the resultant is the same as that obtained by replacing the first l_1 of the X's (real numbers) by their weighted mean and the corresponding weights by their sum (where l_1 is the value for which the weighted mean of the first l of the X's is a maximum) and repeating this process on the X's beyond X_{l_1} until X_n is included. The condition is imposed that no two weighted means shall at any stage be equal.

Secondly it is shown that over all $(n!)^2$ permutations of $X_1, \ldots, X_n$ and of $W_1, \ldots, W_n$ the probability distribution function of k is independent of the particular values of the X's and W's. This is done by induction, a method that differs from that of Sparre Andersen [*Math. Scand.* (1954) **2**, 209-218] who showed the result for equal weights. The proof shows further that the probability density function of k when multiplied by n obeys the recurrence and boundary conditions of the Stirling numbers of the first kind; thus obtaining the probability density function given by Sparre Andersen.

(D. E. Barton)

TORTORICI, P. (Economics Faculty, Univ. of Rome)

1.1 (–.–)

On a theorem relating to the " Probabilities zero and one "—*In Italian*

G. Ist. Ital. Attuari (1959) **22**, 93-98 (9 references)

The author considers an unlimited set of events E_1, E_2, ..., E_i, ... and demonstrates that if the probability of the event consisting in the occurrence of at least one of the events E_h, E_{h+1}, ..., E_k (any h and k whatever) is not smaller than the probability of the occurrence of the same event in the hypothesis that the events E_i are independent of one another, then the probability that, in one trial on the probability design from which the E_i come, an infinite number of events E_i will occur is either zero or one, according as the series $\sum_{i=1}^{\infty} \Pr(E_i)$ (where $\Pr(E_i)$ denotes the probability of the occurrence of E_i in one trial apart from the occurrence or not of the other events) is either convergent or divergent.

As a consequence of this theorem another is deduced, which is due to Franckx, who still with reference to the theorem called " of the probabilities zero and one " had recently extended a theorem by Borel.

Then after introducing the events $0_k^{(k-h)}$, consisting of the occurrence of all the events $\bar{E}_h$, $\bar{E}_{h+1}$, ..., $\bar{E}_{k-1}$ contrary to the events E_h, E_{h+1}, ..., E_{k-1} and denoting by $\Pr(E_k/O_k^{k-h})$ the probability of the occurrence of E_k supposing $\bar{E}_h$, $\bar{E}_{h+1}$, ..., $\bar{E}_{k-1}$ have occurred, the author demonstrates that a necessary and sufficient condition that it be certain that an infinite number of events E_1 will occur is that the series $\sum_{h=k}^{\infty} \Pr(E_k/0_k^{(k-h)})$ shall be divergent for all the values of h.

Finally, simpler conditions are obtained, either " only necessary " or " only sufficient ". So the following are the conditions which are necessary for the occurrence of an infinite number of events E_i to be certain: (1) the series $\sum_{h=k}^{\infty} \Pr(E_k/0_k^{(k-h)})$ shall be divergent for one h, for instance for $h = 1$; (2) it shall not be divergent for all the k's, $1/k \log \Pr(E_k/0_k^{(k-h)}) < a < 0$. As to sufficient conditions there are the following:

(i) that Σp_k^* shall be convergent, p_k^* being
$= \min_{h=1,2,\ldots,k} \Pr(E_k/0_k^{(k-h)})$;

(ii) in particular, that it shall be $p_k^* \geqq \varepsilon > 0$ for an infinite number of k's;
$p_k^* \geqq \varepsilon \Pr(E_k)$, $(\varepsilon > 0)$, definitively;
$p_k^* > \frac{\varepsilon}{k} 1 (\varepsilon > 0)$, definitively.

(A. Del Chiaro)

DARWIN, J. H. (D.S.I.R., Wellington, New Zealand) **2.5** (4.3)

An ecological distribution akin to Fisher's logarithmic distribution—*In English*

Biometrics (1960) **16,** 51-60 (7 references, 3 tables)

This paper is concerned with the derivation of a probability distribution to describe the number of species of animals or plants appearing in n of M areas or time samples. The distribution derived is to be distinguished from Fisher's logarithmic distribution primarily by the fact that the former requires only the determination of which different species appear in a sample, whereas the latter requires the counting of the numbers of each species in the sample, this is sometimes impossible or inconvenient.

Consider M areas or time samples for which one counts the number n of samples containing a particular species, $n = 1, 2, \ldots, M$. For a given species with probability p of appearing in a sample it is assumed that the conditional distribution of the random variable n is binomial with parameters M and p. Further, a hypothetical population of species is postulated with probabilities p following the beta distribution with parameters A and B. The marginal distribution of n is now determined and denoted by $\{P_n\}$, $n = 0, 1, \ldots, M$. This paper restricts attention to results for small values of A only, since for this case intervals of p near zero have the large probabilities.

Suppose a finite random sample of size Z is drawn from the population of species which is to constitute the total sub-population of species available for the M samples. The marginal distribution of n is the same as before, and letting x_n denote the number of species appearing in n of the M samples, $x_n = 0, 1, \ldots, Z$, the marginal distribution of x_n is binomial with parameters Z and p_n, and the joint distribution of $x_1, x_2, \ldots, x_M$ is multinomial. Taking the limit of the joint distribution as Z approaches infinity, A approaches zero, and ZA approaches k, the limiting distribution is found to be the joint distribution of M independent Poisson variables with parameters $\mathscr{E}(x_n) = u_n$, $n = 1, 2, \ldots, M$, as given. In order to estimate the remaining parameters k and B, the author suggests foregoing the method of maximum likelihood in favour of the method of moments in order

DARWIN, J. H. (D.S.I.R., Wellington, New Zealand) **2.5** (4.3) *continued*

An ecological distribution akin to Fisher's logarithmic distribution—*In English*

Biometrics (1960) **16,** 51-60 (7 references, 3 tables)

to eliminate the difficulty of solving the equations involved. The latter leads to the solution of two equations obtained by equating to their expectations two sample quantities, S the number of different species appearing in the M samples and N the sum of the numbers of species appearing in the individual samples. A table is provided to facilitate the solution of the two equations, and it is stated that the estimate of k thus obtained is approximately normally distributed with variance $k/\log(M/B)$ when M and B are large.

Two examples are given in which observed x_n values agree very well with their estimated expectations u_n.

(C. C. Thigpen)

EEDEN, Constance van & RUNNENBURG, J. T. (Mathematical Centre, Amsterdam) **2.4** (5.6)

Conditional limit distributions for the entries in a 2×2 table—*In Dutch*

Statist. Neerlandica (1960) **14**, 111-126

Statist. Dept., Math. Centre, Report S254 [M80*a*] (1960)

In this paper the following problem is treated. Consider, for $\nu = 1, 2, \ldots$, the sequence of two by two tables

$$\begin{array}{cc|c} a_{1,\nu} & a_{3,\nu} & m_\nu \\ a_{2,\nu} & a_{4,\nu} & n_\nu \\ \hline r_\nu & s_\nu & N_\nu, \end{array}$$

where, for each $i = 1, 2, 3, 4$, the $a_{i,\nu}$ has a hypergeometric distribution. Let $\mu_{i,\nu}$ denote the expectation of $a_{i,\nu}$ and let σ_ν^2 denote their variance, then for each ν the symbols m_ν, n_ν, r_ν and s_ν may be assigned to the marginal totals of the two by two table in such a way that

$$\mu_{1,\nu} \leqq \mu_{2,\nu} \leqq \mu_{3,\nu} \leqq \mu_{4,\nu}.$$

Now let $\lim_{\nu\to\infty} N_\nu = \infty$, then the limit-distribution of $a_{1,\nu}$ for $\nu\to\infty$ is considered under the condition and the following statements are proved. In order to simplify the notation the index ν is omitted.

I. The limit-distribution of a_1 exists if μ_1 and σ^2 both have a (finite or infinite) limit for $\nu\to\infty$. Then, with $\lambda \overset{\text{DEF}}{=} \lim_{\nu\to\infty} \mu_1$, $\tau^2 \overset{\text{DEF}}{=} \lim_{\nu\to\infty} \sigma^2$,

(i) if $\tau^2 = 0$, a_1 has asymptotically a degenerate distribution,

(ii) if $0 < \tau^2 < \lambda < \infty$, a_1 has asymptotically a non-degenerate binomial distribution with mean λ and variance τ^2,

(iii) if $0 < \tau^2 = \lambda < \infty$, a_1 has asymptotically a non-degenerate Poisson distribution with parameter λ,

(iv) if $\tau^2 = \infty$, $a_1 - \mu_1/\sigma$ has asymptotically a standard normal distribution.

II. The limit-distribution of a_1 does not exist if $\lim_{\nu\to\infty} \mu_1$ and (or) $\lim_{\nu\to\infty} \sigma^2$ does not exist, i.e. in this case two subsequences $\{\nu'\}$ and $\{\nu''\}$ of the sequence $\{\nu\}$ exist such that the limit-distributions of a_1 for $\nu'\to\infty$ and for $\nu''\to\infty$ are not identical.

As far as the authors know the asymptotic normality of $a_1 - \mu_1/\sigma$ was up to this report only proved under the more stringent conditions of

$$\lim_{\nu\to\infty} r = \infty, \quad \lim_{\nu\to\infty} \inf \frac{m}{N} > 0$$

[see Madow, *Ann. Math. Statist.* (1948) **19**, 535].

(Constance van Eeden)

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GULOTTA, B. (University of Palermo) **2.3** (6.0)

The extension of a theorem by Khintchine over the domain of the normal law—*In Italian*

G. Ist. Ital. Attuari (1959) **22**, 40-58 (5 references)

After considering a two-dimension distribution law $G(x, y)$, the author seeks for sufficient conditions that another distribution law $F(x, y)$ shall belong to the domain of the law $G(x, y)$, i.e. it shall be such that if $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ are n double independent random variables for which the law $F(x, y)$ holds, there shall exist for every natural number n, four numbers $\xi_n > 0$, $\eta_n > 0$, μ_n, ν_n, such that the variate $(X^{(n)}, T^{(n)})$ as defined by the relationship

$$\begin{cases} X^{(n)} = \dfrac{x_1 + x_2 + \ldots + x_n}{\xi_n} - \mu_n \\ Y^{(n)} = \dfrac{y_1 + y_2 + \ldots + y_n}{\eta_n} - \nu_n \end{cases}$$

follows a law $F_n(x, y)$ which approaches $G(x, y)$ for $n\to\infty$.

In this note, a necessary and sufficient rule by Khintchine that a one-dimension distribution law shall belong to the domain of the normal law is extended to two-dimension distribution laws; yet still limited to being sufficient.

If R is the rectangle defined by the limitations $-\alpha > x \leqq +\alpha$, $-\beta < y \leqq +\beta$ and if

$$H_{ij}(\alpha, \beta) = \frac{1}{\alpha^i \beta^j} \int_R x^i y^j dF, \quad \chi(\alpha, \beta) = 1 - H_{00}(\alpha, \beta)$$

it is demonstrated that if

$$\lim_{\substack{\alpha\to\infty \\ \beta\to\infty}} \frac{\chi(\alpha, \beta)}{H_{20}(\alpha, \beta)} = \lim_{\substack{\alpha\to\infty \\ \beta\to\infty}} \frac{\chi(\alpha, \beta)}{H_{02}(\alpha, \beta)} = 0$$

and if there exists the limit

$$r = \lim_{\substack{\alpha\to\infty \\ \beta\to\infty}} \frac{H_{11}(\alpha, \beta)}{\sqrt{H_{02}(\alpha, \beta)\, H_{20}(\alpha, \beta)}}$$

then the law $F(x, y)$ belongs to the domain of the reduced normal law having as a coefficient of correlation

$$\rho = \lim_{\substack{\alpha\to\infty \\ \beta\to\infty}} \frac{H_{11} - H_{10} \cdot H_{01}}{\sqrt{(H_{20} - H_{10}^2)(H_{02} - H_{01}^2)}}.$$

(A. Del Chiaro)

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FILIPPO, E. (Economics Faculty, Univ. of Cagliari) **2.1** (2.8)

On the determination of the probability and intensity of transvariation between two variates in the case of their following a law of distribution due to Edgeworth—*In Italian*

G. Ist. Ital. Attuari (1959) **22**, 82-92 (3 references)

In this work a law of probability is considered, which is obtained (following Pearson) by approximating the binomial polygon by means of a curve obtained by integrating a suitable differential equation.

Precisely, the density of the probability of a variate X following such a law is given by

$$y = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-M)^2}{2\sigma^2}}\left[1+\frac{K}{6}\left(\frac{x-M}{\sigma}\right)^3\right],$$

M being the mode of the distribution; σ^2 the mean value of $(X-M)^2$; $K = 2\,\frac{m-M}{\sigma}$, where m is the mean value of X.

Briefly, the law above is made up by Gauss's normal law with a correcting term, by means of which the possible asymmetry of the binomial curve is taken into account.

Then the author considers two variates X and X', each of which follows the probability law above, and determines for them the transvariability and the sum of the intensities of transvariation and then the probability and intensity of transvariation.

For the transvariability the author obtains the following expression:

$$p = \tfrac{1}{2} - \tfrac{1}{2}\theta(\alpha) + \phi(\alpha)\,\frac{\frac{2}{3}(\alpha^2+1)(K'\sigma'^3 - K\sigma^3) + \sigma^2\sigma'^2\left(\frac{K'}{\sigma'} - \frac{K}{\sigma}\right)}{2\sigma^2+2\sigma'^2}$$

where

$$\alpha = \frac{M-M'}{\sqrt{2\sigma^2+2\sigma'^2}};\quad \phi(\alpha) = \frac{1}{\sqrt{\pi}}\, e^{-\alpha^2};\quad M,\ K \text{ and } \sigma$$

are the parameters defined above for the variate X and M', K' and σ' the similar parameters for the variate X' and it is supposed $\alpha > 0$.

The following expression is obtained for the sum of the intensities of transvariation

$$i = \phi(\alpha)\left(\sqrt{\frac{\sigma^2+\sigma'^2}{2}} + \frac{\alpha}{\sigma}\cdot\frac{K'\sigma'^3 - K\sigma^3}{\sigma^2+\sigma'^2}\right) + \frac{1-\Theta(\alpha)}{4\sigma^2+4\sigma'^2}$$

$$\left[K'\sigma'^3 - K\sigma^3 + \sigma^2\sigma'^2\left(\frac{K'}{\sigma'} - \frac{K}{\sigma}\right) - \alpha(2\sigma^2+2\sigma'^2)^{\frac{3}{2}}\right].$$

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FILIPPO, E. (Economics Faculty, Univ. of Cagliari) **2.1** (2.8) *continued*

On the determination of the probability and intensity of transvariation between two variates in the case of their following a law of distribution due to Edgeworth—*In Italian*

G. Ist. Ital. Attuari (1959) **22**, 82-92 (3 references)

Then for p and i, approximate simpler expressions are obtained, and from these expressions the maximum values, still approximate, in order to obtain from them, by relating to them the values of p and i, the required probabilities and intensities of transvariation.

Finally, the author demonstrates that the expressions known for the calculus of the probability and intensity of transvariation in the case when the two variates follow the normal law are valid, in the first approximation, when the two laws of distribution are asymmetric, provided the two third order moments of the variates in relation to their means are equal.

(A. Del Chiaro)

GUMBEL, E. J. (Columbia University, New York) **2.6** (11.2)
Statistical theory of extreme values—*In English*
Bull. Int. Statist. Inst. (1958) **36**, III 12-13 (2 references)

This theory deals with the largest, smallest and absolute largest values of n independent observations, their range and midrange; either for small samples and a given initial distribution or (asymptotically) for large samples and a given type of initial distribution. The theory of exceedences which is distribution-free gives the probability that an order statistic chosen for a given sample size will be exceeded x times in a future sample of the previous or any other size. The variance of such a prediction is *minimum* for the largest value.

For a given initial distribution the probability function of the largest among n independent observations is the nth power of the initial probability function. Hence all quartiles are known once a table of the median of the largest value as function of n has been calculated. The distribution of the smallest value is obtained from the distribution of the largest value by a symmetry relation. For any continuous distribution the difference between the mean largest value and the initial mean increases slower than $\sqrt{n/2}$ times the initial standard deviation.

The asymptotic theory of extreme values is centred about the analysis of the exponential distribution. The main notion is that of stability: the distribution of the largest value should be the same as the initial one except for a linear transformation of the variate by factors which depend upon the sample size. For the exponential type, the asymptotic probability of the largest value is the double exponential function $\exp[-\exp(-y)]$ where y is a linear transformation of the original variable x. The precision of the largest value need not increase with the sample size from which the largest value was taken. The asymptotic distribution of the midrange is logistic. For an exponential transformation of the range it is a Bessel function of the second kind and of order zero. Finally the distribution of the absolute largest value for n observations is linked to the distribution of the largest value for $2n$ observations.

In addition to the exponential type two other types of initial distributions exist having stable distributions of the extremes. The second one is the Cauchy type and

GUMBEL, E. J. (Columbia University, New York)
Statistical theory of extreme values—*In English*
Bull. Int. Statist. Inst. (1958) **36**, III 12-13 (2 references)

2.6 (11.2)
continued

the third one is a limited type. The three asymptotic distributions are linked by logarithmic transformations. In all cases the estimation of the parameters by maximum likelihood is very complicated, while the method of moments (in the second case for the logarithms) is simple to apply and leads for the third type to estimations of the lower and upper limits for the smallest and largest value respectively. A probability paper leads to a distinction between the three types and may be used for the six distributions of extremes.

(E. J. Gumbel)

GUMBEL, E. J. (Columbia University, New York) **2.0** (6.1)
Multivariate distributions with given margins—*In English*
Rev. Fac. Cienc. Lisboa, A (1959) **7**, 179-218

The author deals with the problem of the construction of bivariate distribution functions with given margins, some related questions and a generalisation. He formulates a general system of distribution functions with given margins, based on a work of Morgenstern. This system has a free parameter, related to the correlation coefficient. The null value of the parameter gives the independence. This system is very different from the convex system based on the limits given by Fréchét to all bivariate distribution functions which have given margins; this system of Fréchét does not contain independence as a special case. The regression lines, the correlation ratios and coefficient are obtained. An interesting result on the regression lines is the following one: those lines on one variable are linear tranforms of the marginal distribution function on the other variable; only in the case of independence are the regression lines straight (orthogonal). Some results are unusual: the correlation coefficient varies between $-1/3$ and $+1/3$, the correlation ratios are equal and the equi-probability curves are not elipses even when the margins are normal.

The author also defines the symmetry and the mutual symmetry of random variables and obtains some of its properties.

A generalisation of these results to multivariate distributions is given together with tables and graphs for some problems.

(J. T. de Oliveira)

2/29

HARRIS, E. K. (Robert A. Taft Sanitary Engineering Center, Cincinnati) **2.4** (4.6)
Analysis of experiments measuring threshold taste—*In English*
Biometrics (1960) **16**, 245-260 (9 references, 5 tables, 1 figure)

The distribution function of individual threshold concentrations in taste perception experiments has been investigated for non-parametric and binomial discrimination models. An individual's threshold is defined to be a concentration X_0, such that for all $X \geqq X_0$, the individual is certain to discriminate exactly between concentrate and control solutions; but for $X < X_0$ correct identification will occur only by chance. An estimate of the cumulative distribution function of the true log threshold is given for a general probability distribution of apparent thresholds. Alternatively, a parametric model is proposed in which each panel member may be characterised by a certain number of test concentrations, n_s, for which discrimination is a matter of chance with probability p, so that the probability of m correct determinations in n_s follows the binomial law. The assumption that n_s itself is a binomial variate yields in turn a binomial distribution for the observable quantity $n_s - m$. In an attempt to improve the fit of this model to threshold data, the binomial parameter is assumed to be a random variable with a gamma or beta density. This more general model is justified by an argument based upon the minimum number of taste-stimulating molecules required for discrimination.

Taste perception experiments with zinc chloride and ferrous sulphate solutions have been analysed by the binomial and non-parametric techniques. Experience has indicated that the former model gives an approximately linear curve on log-normal probability paper, while the non-parametric estimates generally oscillate about that curve. However, confidence intervals for the threshold distribution may usually be more conveniently obtained with the non-parametric method.

For the binomial procedure, a relative confidence-interval width criterion is given for determining the effect of a reduction in the probability of correct discrimination or an increase in panel size.

(D. F. Morrison)

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IHM, P. (Botanic Institute, Freiburg, Germany) **2.3** (11.4)

On the numerical evaluation of the n-dimension normal distribution—*In German*

Metrika (1960) **3**, 74-77

In this paper the author treats the numerical evaluation of a special type of multivariate normal integral, assuming a covariance matrix $\boldsymbol{B}$ of the form $\boldsymbol{B} = b^2\boldsymbol{I}+s\boldsymbol{d}\boldsymbol{d}'$.

Here $\boldsymbol{I}$ denotes the identity matrix, $\boldsymbol{d}^1$ the vector (1, 1, ...1) and s a scalar equal ± 1. The author considers the integral

$$\mathscr{J}_G = \frac{1}{(2\pi)^{n/2}\,|\,\boldsymbol{B}\,|^{1/2}} \int_G \exp\left(-\tfrac{1}{2}z'\boldsymbol{B}z\right)dz_1 \ldots dz_n.$$

The domain of integration G is restricted to be of the type $z_i \geqq c$, $i = 1, 2, \ldots n$.

Previously the author dealt with a more general case with covariance matrix

$$\boldsymbol{A} = \boldsymbol{D} + \frac{1}{c^2}\,\boldsymbol{d}\boldsymbol{d}_1'$$

$\boldsymbol{D}$ being a positive definite diagonal matrix, $c^2 > 0$. [see *Sankhyā* (1959) **21**, 363-366: abstracted in this journal No. 356, **1.0**]. Applying the results obtained there to the special case stated above, the author reduces the n-dimensional integral to the simple integral

$$\mathscr{J}_G = \frac{1}{\sqrt{2\pi}} \int_{\frac{c\sqrt{n^1}}{\sqrt{b^2+s_n}}}^{\infty} \exp\left(\frac{-t^2}{2}\right) N_n\left(t\sqrt{b^2+s_n} - c\sqrt{n}\right)dt.$$

In this formula, $N_n(n)$ means the distribution function of the extreme deviate from the arithmetic mean, tabulated by Nair [*Biometrika* (1948) **35**, 118-144]. Using Stieltjes integral co-ordinates, the latter integral may be evaluated easily.

(J. Roppert)

JOHNSON, N. L. (University College, London) **2.4** (2.7)

An approximation to the multinomial distribution: some properties and applications—*In English*

Biometrika (1960) **47**, 93-102 (12 references)

The approximation to the multinomial distribution which is obtained by regarding the relative cell or class frequencies as continuous variables may be considered to be a generalisation of the Pearson Type I curve fitted to a binomial distribution. The approximation gives the correct range of variation, the correct first and second moments and product-moments of order two. Although the skewness of the distribution of a continuous variable is twice as large as that for the discrete variable and the kurtosis is almost six times as large, this does not appear to reflect in any serious difference between the true and approximate frequencies.

Given that the approximation is adequate, the author indicates some useful analogies; for example

(i) the ratio of the frequencies in two classes of the multinomial distribution is distributed as an F-ratio of two independent chi-squared variates.

(ii) if the classes of the multinomial distribution are assigned fixed numerical values then the distribution of the rth crude sample-moment is as the value of a weighted sum of independent chi-squared variates to the unweighted sum of the same chi-squared variates.

Although the approximation is consistent in that the distribution remains of the required form, if one or more of the variables is integrated out, the consistency does not extend to the conditional distribution of some of the relative cell frequencies given other cell frequencies.

Reference is made to the possibility of using the Bartlett-Neyman-Pearson test for equality of variance [Bartlett: *Proc. Roy. Soc. A* (1937) **160**, 268-282 and Neyman & Pearson, E. S.: *Bull. Acad. Polon. Sci.* (1931) 460-481] as a means of testing the hypothesis that the expected cell frequencies are equal.

Finally, the author notes that the maximum likelihood estimates of the class probabilities in the continuous and discrete cases differ, though this difference is small in expectation, being inversely proportional to the sum of the cell frequencies.

(J. G. Saw)

MAULDON, J. G. (Corpus Christi College, Oxford)
A generalisation of the beta-distribution—*In English*
Ann. Math. Statist. (1959) **30**, 509-520 (7 references)

The author defines and studies a class of distributions which include as special cases, the ordinary beta-distribution, the triangular distribution, the rectangular distribution over any non-degenerate simplex, and a continuous range of other distributions over such a simplex called basic β-distributions, which are analogous to the ordinary β-distributions. Various other distributions which arise in connection with the random division of an interval are also included in the class studied.

The author begins by letting $x^1, \ldots, x^n$ be n random variables with the joint distribution function $F = F(x^1 \ldots, x^n)$. He introduces a certain integral transform of this distribution defined as

$$\phi_p(t; a_1, \ldots, a_n) = \mathscr{E}\left\{\left(t - \sum_{j=1}^{n} a_j x^j\right)^{-p}\right\}, \quad (p>0),$$

where p is the exponent of the transformation, and where t and $a_j (j = 1, \ldots, n)$ are the $n+1$ (homogeneous parameters of the distribution. The following main results are proved:

Theorem 1. For any given set of $r+rn$ real constants, $p_i > 0$) and $c_i^j (i = 1, \ldots, r;\ j = 1, \ldots, n)$, there is exactly one joint distribution of the variates $x^1, \ldots, x^n$ which satisfies the relation

$$\phi_p(t; a_1, \ldots, a_n) = \prod_{i=1}^{r}\left(t - \sum_{j=1}^{n} a_j c_i^j\right)^{-p_i},$$

where

$$p = \sum_{i=1}^{n} p_i.$$

Theorem 2. Let Δ be the convex hull of the r points whose vector co-ordinates are c_i, suppose that Δ is of dimension d $[1 \leqq d \leqq \min(r-1,\ n)]$, and let δ be the d-dimensional hyperplane containing Δ. Then any β-distribution with vertices c_i is a continuous distribution over δ with positive d-dimensional density at all interior points of Δ and zero density elsewhere.

Further if $p_i \geqq 1$ (all i) the density of the distribution is bounded, and if $p_i > 1$ (all i) the density is continuous over δ.

Theorem 3. Any β-distribution which is not concentrated at a single point admits only one possible value for its exponent.

MAULDON, J. G. (Corpus Christi College, Oxford)
A generalisation of the beta-distribution—*In English*
Ann. Math. Statist. (1959) **30**, 509-520 (7 references)

Corollary. Any β-distribution which is not concentrated at a single point has a unique set of distinct vertices, and its corresponding indices are also uniquely determined.

The paper also contains three more theorems, six lemmas and corollaries together with some discussions and examples of the basic β-distribution.

(A. C. Cohen, Jr.)

PISTOIA, A. (Math. Inst., Milan Polytechnique) **2.1** (–.–)

On an observation about characteristic functions—*In Italian*

G. Ist. Ital. Attuari (1959) **22**, 79-81 (2 references)

In this note the following are introduced:

(1) Class S of sequences $\{\alpha_k\}$ of non-negative numbers, for which $\sum_{k=0}^{\infty} \alpha_k = 1$

(2) Class R of distribution functions, i.e. of the functions $F(t)$ never decreasing, continuous on the right, and for which:

$$\begin{cases} F(-\infty) = \lim_{t \to -\infty} F(t) = 0 \\ F(+\infty) = 1 \end{cases}$$

(3) Class C of characteristic functions $f(x)$ defined by

$$f(x) = \int_{-\infty}^{+\infty} e^{itx} dF(t)$$

$F(t)$ being a function of class R.

Considering then a sequence of functions $F_k(t)$ belonging to R, the corresponding sequence of characteristic functions

$$f_k(x) = \int_{-\infty}^{+\infty} e^{itx} dF_k(t)$$

and the series $\Psi(x) = \sum_{k=0}^{\infty} \alpha_k f_k(x)$, where the coefficients α_k belong to S, the author demonstrates that $\Psi(x)$ belongs to C, that $H(t) = \sum_{k=0}^{\infty} \alpha_k F_k(t)$ belongs to R and finally that between the functions $\Psi(x)$ and $H(t)$ there exists the relationship

$$\Psi(x) = \int_{-\infty}^{+\infty} e^{itx} dH(t).$$

That is to say $\Psi(x)$ is the characteristic function corresponding to the distribution function $H(t)$.

(A. Del Chiaro)

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PRINS, H. J. (Philips Research Laboratories, Eindhoven) **2.2** (11.1)

Transforms for finding probabilities and variate values of a distribution function in tables of a related distribution function—*In English*

Statist. Neerlandica (1960) **14**, 1-17 (14 references, 3 tables)

The main purpose of this article is to bring together in a convenient form, and ready for immediate use, the transformations by which the various distribution functions currently used in statistical problems are interconnected. The main results are presented in two tables. Table I refers to the normal, χ^2-, gamma- and Poisson distributions and presents in a two-entry array the transformation formulae for the parameters, the variate, and the cumulative probability by which these distributions can be transformed into one another. For example, to transform the gamma-distribution into the χ^2-distribution we must put

$$2(p+1) = \nu,\ 2u\sqrt{p+1} = \chi^2,\ 1-P(u<u_0) = P(\chi^2>\chi_0^2).$$

Table II contains the cross-tabulations of the transformations for the t-, F-, G-, r-, B-, binomial, and negative binomial distributions. Here the G-distribution is the distribution of the ratio of the means of two samples from the same exponential distribution.

The same symbols are used throughout as in the most extensive tables at present available. The accompanying text contains a brief description of each of the distributions considered, and gives some examples of the use of the main tables.

Through a misprint, the information that to transform the B-distribution into a t- or r-distribution we must put $q = 1/2$, is missing in Table II.

(H. C. Hamaker)

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STEYN, H. S. (Univ. of South Africa, Pretoria) **2.7** (3.5)

On regression properties of multivariate probability functions of Pearson's types—*In English*

Proceedings Kon. Ned. Akad. Wetensch. A (1960) **63**, 302-311.
Indag. Math. (1960) **22**, 302-311

This paper refers to a theorem proved by the author in an earlier paper [*Indag. Math.* (1957) **19**, 119-127] where a necessary and sufficient condition is derived that for a multivariate probability function $f(x_1, x_2, \ldots, x_k)$ the regression function of x_1 on the other variables is of the form Q/R where Q and R are polynomials in $x_2, x_3, \ldots, x_k$. The condition referred to above is in terms of differential operators operating on the moment generating function.

The author now rewrites this theorem for the case of a continuous multivariate probability density function and uses it to derive the linear regression equations for the multivariate normal probability function and a multivariate Beta-type function. He then considers a system of differential equations in $f(x_1, x_2, \ldots, x_k)$ such that the solutions will give a generalised Pearsonian system of frequency functions all with linear regression equations.

Proceeding to multivariate continuous functions with non-linear regression functions of the form Q/R he derives the regression function for the well-known probability density function of the sample correlation coefficients r_{ij} for the particular case where the k-variables in the normal parent distribution are independent. A generalised Pearsonian system of continuous probability functions is then considered and shown to have rational regression equations (e.g. the regression of x_1 on $x_2, \ldots, x_k$ is given by an equation $\hat{x}_1 = Q(x_2, \ldots, x_k)/R(x_2, \ldots, x_k)$, where Q and R are again polynomials). As a special case the regression functions for the well-known Wishart distribution are discussed. If $\mathbf{V} = [v_{ij}]$ is the moment matrix for a sample from a k-dimensional normal distribution, then the regression function of V_{ii} is found as a rational function of the other second-order sampling moments. For a non-diagonal element v_{ij} the regression function of v_{ij} is derived only for the special case when in the normal parent population the variables x_i and x_j are independent.

(H. S. Steyn)

2/37

WEILING, F. (Inst. Landw. Bot., Univ. Köln, Germany) **2.2** (7.8)

The use of transformations in experimental analysis, with special reference to ranking in an arbitrary interval—*In German*

Der Züchter (1959) **29**, 281-284 (14 references, 3 tables)

Well-known pre-requisites for the applicability of the standard analysis of variance on experimental and similar data are normal distribution of the variates, homogeneity of variances (within treatment mean squares) and additivity of treatment effects. Failure of this model assumptions can be overcome by means of an appropriate transformation of variates. In this paper a special case is considered where the experimental values (from plant-breeding experiments) are ranks x defined in the arbitrarily fixed closed interval $a \leqq x \leqq b$ (here: $a = 0, b = 4$).

An appropriate transformation $y(x)$ for stabilisation of the variances and removal of non-additivity was achieved by:

(i) transforming the ranks x to logarithmic fractions p defined in the interval $0 \leqq p \leqq 1$ according to $p = \dfrac{\log(b+1-x)}{\log(b+1-a)}$ and,

(ii) applying the probit-transformation $y = \text{probit}\,(p)$ in the example considered.

This double transformation effectively removes heterogeneity of variances and non-additivity of the original ranked data and the analysis of variance performed on the transformed ranks gives different results compared to those obtained from the original data.

(R. Wette)

2/38

LUKÁCS, E. (Catholic University of America, Washington) **3.8** (4.6)

On distribution-free partition statistics for the normal family—*In English*

Bull. Int. Statist. Inst. (1958) **36**, III 37-42 (2 references)

In the introduction the author states that there are cases where a statistic S is dependent only on the sample of observations and remains unaffected by the choice (or otherwise) of population distribution. These statistics may be called " partition " statistics because they can be represented by partitions of the sample space. After noting that the concept of distribution-free statistics was introduced by Z. W. Birnbaum [*Ann. Math. Statist.* (1953) **24**, 1-8] the author of this present paper states that its object is to give the characterisation of the partition statistics which are distribution-free in the normal family. The problem of the characterisation was suggested by Birnbaum and the results depend on a lemma concerning the condition for a random variable Y to possess a constant regression on a second random variable X: this lemma is stated in the second section of the paper.

The third section assumes the family of distribution functions consisting of all normal distributions and also the one-parameter sub-families obtained by fixing either of the two parameters in the normal distribution. After writing down an expression for the characteristic function of the partition statistic S the author notes that this will be distribution-free in the normal family if and only if the two first partial derivatives—one with respect to μ and the other with respect to σ—are zero.

Theorem 1 states the conditions, relating to constant regression on S, for the partition statistic S to be distribution-free in the two sub-families. Theorem 2 states the conditions for the whole normal family. The first corollary to a more general theorem (No. 3) assumes existence of distribution-free statistics for the whole normal family and the second corollary does this for the two sub-families.

(W. R. Buckland)

SAW, J. G. (University College, London) **3.8** (–.–)

A note on the error after a number of terms of the David-Johnson series for the expected values of normal order statistics—*In English*

Biometrika (1960) **47**, 79-86 (4 references)

Given n normal variates, it is possible, using the David-Johnson technique [*Biometrika* (1954) **41**, 228-240], to approximate to the expected value of the rth smallest variate by means of the first few terms of an infinite series in $1/N$, the coefficients in which are functions of $r/(n+1)$ only. The series appears to converge quite rapidly when compared with known exact results. However, it is desirable to have bounds for the error after a given number of terms of the series in order to compare more fully the David-Johnson method with other methods which have been proposed.

It is found that the error, after a number of terms of the series, may be expressed as a double integral over a region of finite area. The integrand is the product of two terms, one of which is unbounded although the integrand as a whole remains bounded. An investigation into the inverse derivatives of the normal integral indicates how the integrand which contains such inverse derivatives, may be split into factors which are bounded within the region of integration and in this way obtain upper and lower bounds to the double integral.

In the last section, comparison is made with the method proposed by Plackett [*Ann. Math. Statist.* (1958) **29**, 131-142]. It seems that, of the resulting series, Plackett's converges slightly more rapidly and the bounds for error which Plackett is able to give for his series are slightly finer than those which are given for the David-Johnson series. However, it is pointed out that the David-Johnson series finds advantage in the facility of its computation.

(J. G. Saw)

SIBUYA, M. (Institute of Statistical Mathematics, Tokyo) **3.8** (3.9)

Bivariate extreme statistics, I—*In English*

Ann. Inst. Statist. Math., Tokyo (1960) **11**, 195-210 (4 references, 2 figures)

This paper treats the joint limit distributions of maxima of each component in bivariate random samples.

If the probability Pr $\{X>x,\ Y>y\}$ is rather smaller than Pr $\{X>x\}$, Pr $\{Y>y\}$, X_{max} and Y_{max} are asymptotically independent. Conversely, if Pr $\{X>x,\ Y>y\}$ is near to Pr $\{X>x\}$ and Pr $\{Y>y\}$, $(X_{max},\ Y_{max})$ degenerates asymptotically on a curve. Contrary to the univariate case, the possible types of limit distributions are various; their explicit form includes an arbitrary function with some restrictions.

Throughout the discussions, attention is focussed on the dependence between marginal distributions, as their behaviour is well understood. For this purpose, the "dependence function", the ratio of bivariate distribution function to the product of marginal distributions, is introduced and discussed.

(M. Sibuya)

2/41

ZIA-UD-DIN, M. (Panjab University, Lahore, Pakistan) **3.0** (2.1)

On expression of the k-statistic (k_{11}) in terms of power sources and sample moments—*In English*

Bull. Int. Statist. Inst. (1958) **36**, III 102-106 (7 references)

This paper extends an earlier paper of the author when he dealt with (k_9) and (k_{10}) [*Ann. Math. Statist.* (1954) **25**, 800-803]. After referring briefly to the origin and the use of symmetric functions—particularly in the theory of sampling distributions, the author notes the following types:

(i) elementary symmetric functions
(ii) complete homogeneous (or aleph) symmetric functions
(iii) monomial symmetric functions
(iv) power sum symmetric functions
(v) augmented symmetric functions

and that tables have been published of types (iii) and (v) in terms of type (iv).

Whereas for (k_9) and $k_{10})$ the relevant cumulants were written in terms of the moments μ'_r, the expression for (k_{11}) has been worked out by a different method. Using the tables of generalised k-statistics published by Abdel-Aty [*Biometrika* (1954) **41**, 253-260], the expression for (k_{11}) has been written in terms of the power sum products (s_r). By using some simple relationships it was also possible to express (k_{11}) in terms of the sample moments (m_r). [See also abstract of paper by same author, No. 393, **3.0**]

(W. R. Buckland)

2/42

BARANKIN, E. & KATZ, M., Jr. (University of California, Berkeley) **4.0** (4.1)
Sufficient statistics of minimal dimension—*In English*
Sankhyā (1959) **21**, 217-246 (21 references)

In this paper the general problem of finding the smallest number of continuously differentiable, real-valued functions which can constitute a sufficient statistic for a family of probability measures on the Lebesgue measurable subsets of any finite dimensional Euclidean space Ω is discussed. It is assumed that the family of distributions is indexed by a parameter varying over a sub-set of another finite dimensional Euclidean space and every distribution of the family has a density which is continuously differentiable with respect to the parameter as well as the variable entering in the density.

After introducing the notion that a function T on Ω is of dimension r at a point x^0 if there is a neighbourhood of x^0 such that T maps this neighbourhood into a Euclidean r-space, the authors pose the following question: what is the smallest integer r for a given point $x^0 \in \Omega$ such that there exists a sufficient statistic for the given family of distributions, which is of dimension r at x^0 and continuously differentiable in x^0 ? A lower bound is obtained for the dimension of such a sufficient statistic and for " regular points " a method of constructing the dimensionally minimal sufficient statistic with the required differentiability properties is given. It is proved constructively that there exists a sufficient statistic T^* which is of minimal local dimension and " regular " almost everywhere in the set of " regular points ". The existence of a continuously differentiable sufficient statistic with constant dimension equal to the maximum of local dimensions is also shown. Examples are constructed to show that local minimal dimension of a sufficient statistic may vary over Ω.

(K. R. Parthasarathy)

2/43

BASU, D. (Indian Statistical Institute, Calcutta) **4.0** (4.1)
The family of ancillary statistics—*In English*
Sankhyā (1959) **21**, 247-256 (8 references)

An ancillary statistic is one whose sampling distribution is the same whatever parent distribution provides the sample of observations. It is, therefore, not useful in drawing an inference about the distribution from which the sample is drawn. This paper develops a method of classifying ancillary statistics based on the concept of " inclusion ", which is as follows: one ancillary statistic is said to " include " another if the latter can be expressed as a function of the former.

If two ancillary statistics are such that each one can be expressed as a function of the other they are said to be equivalent. A maximal ancillary is defined as being one that cannot be expressed as a function of a non-equivalent ancillary. Given any ancillary statistic, it is shown that there exists at least one maximal ancillary which includes the given one. A minimal ancillary is one that is included by every maximal ancillary. Among the class of minimal ancillaries there always exists a unique one such that every other minimal ancillary is a function of that ancillary. It is termed as the laminal ancillary.

The following properties of laminal, minimal and maximal ancillaries are proved in the paper.

(i) An ancillary and a minimal ancillary together are still ancillary (jointly).
(ii) If a completely sufficient statistic and an ancillary are jointly equivalent to the whole sample then the ancillary must be a maximal ancillary.

It is also proved that under certain conditions the laminal ancillary is essentially equivalent to a constant. In proving these properties the author uses the properties of sub-sigma fields induced by a statistic. The author also gives a number of examples to illustrate the propositions. In one such example the author demonstrates that a family of ancillary statistics can jointly be equivalent to the whole sample.

(Y. R. Rau)

BHATTACHARYA, N. (Indian Statistical Institute, Calcutta) **4.3** (8.6)

An extension of Hald's table for the one-sided censored normal distribution—*In English*

Sankhyā (1959) **21**, 377-380 (2 references, 1 table)

This paper gives an extension of Hald's table for the maximum likelihood estimation of the parameters of a one-sided censored normal distribution. Suppose there are n observations from a normal distribution with mean ξ and standard deviation σ and that a of these observations are less than or equal to a point of truncation. Let the observations falling above the truncation point be $x_1, x_2, \ldots, x_{n-a}$. Hald's method of estimation of ξ and σ involves the evaluation of a certain function $Z = f(h_2 y)$, where h is the fraction a/n of censored observations in the sample and

$$y = (n-a)\sum_{i=1}^{n-a} x_i^2 \Big/ 2\left(\sum_{i=1}^{n-a} x_i\right)^2.$$

He tabulates the values of Z for $h = 0{\cdot}05, 0{\cdot}10, \ldots, 0{\cdot}80$ and for some appropriate values of y. In certain cases of fitting one-sided censored normal distribution to grouped data, h is found to be less than 0·05. Although when h is small, no great loss of information is incurred by ignoring the censored part, it is desirable to consider it for tests of goodness-of-fit. In this paper, the author extends Hald's table to values of h less than 0·05.

The paper contains a brief description of Hald's method of estimation of parameters. In the table, the author gives values of Z for $h = 0{\cdot}001$, 0·010, 0·020, 0·035, 0·050 and $y = 0{\cdot}500$ (·005) 0·600, 0·610 (·010) 1·000, 1·050 (·050) 1·500.

The author remarks that even though $h = 0{\cdot}001$ is outside the range of practical interest, Z is tabulated for that value also because it may aid in interpolation for cases such as $h = 0{\cdot}005$ or 0·008, which are not uncommon.

(Y. R. Rau)

2/45

COHEN, A. C., Jr. (University of Georgia, Athens, U.S.A.) **4.3** (2.8)

Estimating the parameter in a conditional Poisson distribution—*In English*

Biometrics (1960) **16**, 203-211 (12 references, 2 tables, 2 figures)

Maximum likelihood estimators for truncated and censored Poisson samples in both singly and doubly restricted cases have been derived by the author in an earlier paper [*J. Amer. Statist. Ass.* (1954) **49**, 158-168]. More recently, several authors have considered various other estimators because of the bias and more particularly because of the burdensome calculations associated with the maximum likelihood estimators. In this paper the problem of estimating the Poisson parameter when zero values cannot be observed is reconsidered.

The author contends that bias in the maximum likelihood estimators is unlikely to be a major source of trouble in estimating the Poisson parameter except in small samples. The need for large samples is stressed on the basis that the sampling error associated with very small samples will limit the usefulness of the results regardless of the estimator used. The advantage of the consistency and asymptotic efficiency of maximum likelihood estimators is also cited.

The present paper provides tables and charts for simplifying the determination of the estimate of the Poisson parameter and of the variance of this estimate. Table 1 lists values of the estimate corresponding to values of the sample mean. The sample mean is tabled in graded steps from 1·0005 to 12·5. Linear interpolation between the tabled values is claimed to yield at least three significant digits in the results. A folded scale chart graphs δ vs. $\bar{x}$ where $\delta = \bar{x} - \lambda$. When some sacrifice of accuracy is permissible, the parameter is quickly estimated by reading $\hat{\delta}$ from the chart and computing $\hat{\lambda} = \bar{x} - \hat{\delta}$.

The variance of the estimate is approximated by the relation $V(\hat{\lambda}) \doteqdot \Psi(\hat{\lambda})(\hat{\lambda}/n)$. The function $\Psi(\lambda)$ is tabled and charted so that the approximate variance is readily obtained. Several illustrative examples are given.

(M. F. Peck)

2/46

COHEN, A. C., Jr. (University of Georgia, Athens, U.S.A.) **4.3** (2.5)

Estimation in the Poisson distribution when sample values of $c+1$ are sometimes erroneously reported as c—*In English*

Ann. Inst. Statist. Math., Tokyo (1960) **11**, 189-193 (2 references, 1 table)

The author considers a problem of erroneous observations in the Poisson distribution in which values of $c+1$ are sometimes incorrectly reported as c, where c may be any value of the random variable. One of practical examples of this problem arises as that of mis-classification in inspections which involve the counting of defects.

Let the probability of mis-classifying items which actually contain $c+1$ defects as having only c defects be θ. Then the values of the probability function of random variable x, the number of defects reported per unit, are the same as those of the original Poisson distribution for $x = 0, 1, \ldots, (c-1), (c+2), \ldots$, but slightly changed (proportionally to θ) for $x = c$ and $x = c+1$. Using this probability function the author derives the maximum likelihood estimates of θ and the Poisson parameter λ. The required estimates may not exist for some special samples, but the author asserts that when the sample size is large such samples will be very improbable and that their actual occurrence should suggest that the assumed probability function is not applicable to the situation under inspection.

Sampling errors of estimates are evaluated by calculating the asymptotic variance-covariance matrix of the estimates $(\hat{\lambda}, \hat{\theta})$.

At the end of the paper the author illustrates an experimental tabulation of defects per unit reported in an inspection of 1000 units, with the acceptance number, c, being 3. The values of the estimates $\hat{\lambda}$ and $\hat{\theta}$ which were calculated from the data are compared with the corresponding population parameters which were employed in constructing the data.

(K. Isii)

2/47

CORNISH, E. A. (CSIRO, Division of Math. Statist., Adelaide) **4.8** (7.7)

Fiducal limits for parameters in compound hypotheses—*In English*

Aust. J. Statist. (1960) **2**, 32-40 (9 references, 1 figure)

The author's purpose in this paper is "to give the simultaneous fiducial distribution of any number of parameters, based on the joint distribution of a set of t-statistics, which are not only dependent through the common estimate of the variance, but also correlated because of the nature of the enquiry under consideration."

Let $\boldsymbol{X}$ be a column vector of elements $x_1, \ldots, x_N$, normally and independently distributed with means $\xi_1, \ldots, \xi_N$, and common variance σ^2, and let

$$Y = \boldsymbol{XH} = \{h_{ij}\}\boldsymbol{X} \quad (i = 1, 2, \ldots, p;\ j = 1, 2, \ldots, N)$$

define a set $y_1, \ldots, y_p$ of $p<N$ linearly independent linear functions of the x_i, with a vector $\boldsymbol{A} = \{a_i\}$ of means. Then the simultaneous fiducial distribution of the parameters $\{a_i\}$ is obtained as

$$\frac{\Gamma[\frac{1}{2}(n+p)]}{(\pi n)^{\frac{1}{2}p}\Gamma(\frac{1}{2}n)} \left\{1 + \frac{(Y-A)'(s^2\boldsymbol{HH}')^{-1}(Y-A)}{n}\right\}^{-\frac{1}{2}(n+p)} \cdot \frac{dA}{|s^2\boldsymbol{HH}'|^{\frac{1}{2}}},$$

where s^2 is an estimate of σ^2 on n degrees of freedom.

Three important types of matrix $\boldsymbol{H}$ occur in practice, and the appropriate form of the distribution of $\boldsymbol{A}$ is stated for each case.

Several examples illustrating the use of these results are given, and the paper concludes by a comparison with the methods of Dunnett [*J. Amer. Statist. Ass.* (1955) **50**, 1096-1121] and Scheffé [*Biometrika* (1953) **40**, 87-104].

(J. Gani)

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DIXON, W. J. (University of California, Los Angeles) **4.3** (8.6)
Simplified estimation from censored normal samples—*In English*
Ann. Math. Statist. (1960) **31**, 385-391 (4 references, 5 tables)

Estimators of the mean and standard deviation for censored normal samples which are based on linear systematic statistics using simple coefficients and which are almost as efficient as estimators using the best possible coefficients are given for samples of size $N \leqq 20$ for censoring at one extreme and for several types of censoring at both extremes.

More specifically if $x_1 < x_2 < \ldots < x_N$ denote the ordered observations then for the case of symmetrical censoring of i observations at each of the extremes, the estimator $m_w = [(i+1)x_{i+1} + x_{i+2} + \ldots + x_{N-i-1} + (i+1)x_{N-i}]/N$ of the mean has a minimum relative efficiency of 0·99912 for $N \leqq 20$ (this occurs for $N = 20, i = 4$) when compared with the best linear systematic statistic, as given by Sarhan & Greenberg in one of the references. Efficiency is defined as the ratio of the variance of the best linear systematic statistic divided by the variance of the estimator m_w. These efficiencies are tabulated for $N \leqq 20$ and $i \leqq 6$. When one more observation is censored from one extreme than from the other extreme the procedure of dropping another observation to symmetrise the censorship and use m_w as the estimator of the mean is proposed. For $N \leqq 20$, $i \leqq 6$ the efficiency of m_w relative to the best linear systematic statistic is never less than 0·962. When censoring occurs entirely at one extreme, two estimators of the mean and their efficiencies relative to the best linear systematic statistic are compared. The first is the use of m_w after symmetrising the censoring. Its efficiency is never less than 0·956. The second is the estimator $m_a = [ax_1 + x_2 + \ldots + (i+1)x_{N-i}]/(N+a-1)$ where a is chosen so that $E(m_a) = \mu$. m_a is slightly more efficient than m_w and allows estimation of the mean for smaller samples than is possible if one uses m_w. Tables of a and the efficiencies of m_w and m_a relative to the best linear systematic statistic are given.

For symmetrical censoring the use of either the maximum range or a combination of two ranges,

$$(x_{N-i} - x_{i+1}) + (x_{N-i-1} - x_{i+2})$$

or

$$(x_{N-i} - x_{i+1}) + (x_{N-i-2} - x_{i+3}),$$

has minimum relative efficiency of 0·965 for estimation

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DIXON, W. J. (University of California, Los Angeles) **4.3** (8.6)
Simplified estimation from censored normal samples—*In English* *continued*
Ann. Math. Statist. (1960) **31**, 385-391 (4 references, 5 tables)

of the standard deviation. For almost symmetrical censoring, the maximum range or a combination of two ranges,

$(x_{N-i} - x_i) + (x_{N-i} - x_{i+1})$ or $(x_{N-i} - x_i) + (x_{N-i-1} - x_{i+2})$

has the minimum relative efficiency of 0·965. For censoring entirely at one extreme an estimator of the standard deviation based on an optimum choice of one or two ranges has the minimum relative efficiency of 0·937. These estimators and their efficiencies are tabulated. The optimum solution for most cases requires the use of an extreme observation at the censored end with doubled weight rather than two different observations.

(J. W. Wilkinson)

2/50

GREENBERG, B. G. & SARHAN, A. E. (University of North Carolina, Chapel Hill) **4.6** (8.6)

Some applications of order statistics—*In English*

Bull. Int. Statist. Inst. (1958) **36**, III 172-183 (25 references, 6 tables)

The purpose of this expository paper is to show how some recent developments in order statistics help with problems of estimation. Four situations are covered and the paper begins with a definition of order statistics as used by the authors.

(i) " *Censored* " *observations.*

The term " censored " is applied to instances where the exact magnitude of specific observations on a sample may not be known. However the *number* of censored observations is known as is their order, or rank, relative to the censoring point. Censoring is different from truncation, the latter term implies a restriction on the population from which the sample is drawn. For example, military personnel have to pass qualifying height limits, hence a sample of height measurements would be truncated. The main difference between the two terms is that with a censored sample the number of censored items is known, whereas with a truncated population the number of " missing " values is not.

The example given is concerned with the experimental innoculation of twelve mice. On the 64th day after innoculation eight had died. The experimenter wished to estimate the mean and standard deviation from the known eight lives out of the twelve.

The method used is that of " weighting " the known lives by specific factors. These factors are given in full by Sarhan & Greenberg [*Ann. Math. Statist.* (1958) **29**, 79-105].

(ii) *Simplified statistics.*

It is pointed out that the deliberate ommission of data, for purposes of reducing the amount of computation, can still lead to reasonably efficient estimates. For example, using only two-thirds of the sample values, estimates of the mean can be 92·25 per cent. efficient. The detailed example given concerns the distribution of time intervals between explosions in mines [see *Biometrika* (1952) **39**, 168-180]. The mean of the distribution (negative exponential) is estimated from five selected values out of the original 109 observations. The relative efficiency is 94·76 per cent.

GREENBERG, B. G. & SARHAN, A. E. (University of North Carolina, Chapel Hill) **4.6** (8.6) *continued*

Some applications of order statistics—*In English*

Bull. Int. Statist. Inst. (1958) **36**, III 172-183 (25 references, 6 tables)

(iii) *Grouping.*

The problem discussed here is that of grouping a continuous variate into k groups: the central value of the group then represents its members. A common occurrence of this is the familiar grouped frequency table. A short table is given showing the efficiency for $k = 2(1)6$ for the standard normal deviate.

(iv) *Order statistics in short cut tests.*

In this section the authors demonstrate the use of the mean range for estimating the analysis of variance table. The example quoted shows the yields of six varieties in four randomised blocks. A full description is given and reference made to an original paper by Hartley [*Biometrika* (1950) **37**, 271-280].

(A. N. James)

GREENBERG, B. G. & SARHAN, A. E. (Report of discussion on a paper by) **4.6** (8.6)
Some applications of order statistics—*In English*
Bull. Int. Statist. Inst. (1959) **36**, I 128-131

Those taking part in the discussion were Finney, Kulldorf, Bodmer and Miss Sitgreaves: the author briefly replied.

Finney suggested that the methods of the type presented in the paper, and also non-parametric methods, were of interest for three reasons:

(i) purely as a mathematical problem;
(ii) on account of serious doubt about the validity of a parametric model;
(iii) speed of computation where this was economically important in relation to some loss of statistical information.

Kulldorf restricted his comments to section 4 of the paper (grouping) because the results presented there were closely connected with some of his own recent work. Instead of dealing with group central values his criterion was to minimise the asymptotic variance of the maximum likelihood estimate of the mean (μ) [see *Metrika* (1959) **2**, 94-99; abstracted in this journal No. 402, **4.3**].

Bodmer raised the problem of having to use an arbitrary scoring system and its effect upon the grouping of the underlying continuous variable. This problem was the reciprocal one to that considered in the paper. Miss Sitgreaves had also encountered similar situations: for example, where it was possible to know only the interval in which a value of the continuous variable occurred.

(W. R. Buckland)

DURBIN, J. (London School of Economics) **4.3** (4.1)
A note on the application of Quenouille's method of bias reduction to the estimation of ratios—*In English*
Biometrika (1959) **46**, 477-480 (3 references)

Quenouille [*J. R. Statist. Soc.* B (1949) **11**, 68-84 and *Biometrika* (1956) **43**, 353-360] considered an estimator t_1 based on a sample of n observations whose bias had the form $cn^{-1}+O(n^{-2})$, where c is a constant independent of n. The sample is divided randomly into two equal halves, the same estimator computed for each half (taking values t_2, t_3) and the function

$$t = 2t_1 - \tfrac{1}{2}(t_2+t_3)$$

computed. This has bias $\theta(n^{-2})$.

In the present note, the estimator considered has the nature of a ratio of two sample means. This is formalised: x is normal, with unit mean and variance $O(n^{-1})$; conditionally on x, y has mean $\alpha+\beta x$ and variance $O(n^{-1})$; and $r = y/x$ estimates $\alpha+\beta$. Using the " statistical differentials " technique it is shown that not only does Quenouille's device remove the first order term in the bias but also reduces the second order term in the variance leaving the first unaltered. In a second example, of the same regressive structure, but where x is a gamma variable of parameter n, exact results are obtained for the means and variances of r and Quenouille's modification of it and the latter is shown to have smaller mean square error. Finally, a brief numerical comparison is made between Quenouille's method and some methods developed by Goodman and Hartley in relation to a problem treated by the latter. Quenouille's method is found to have a mean square error considerably smaller than any of the four estimators proposed by Goodman & Harley [*J. Amer. Statist. Ass.* (1958) **53**, 491-508].

(D. E. Barton)

ISIDA, M. & TAGAMI, S. (Tokushima University, Japan) **4.1** (2.8)

The bias and precision in the maximum likelihood estimation of the parameters of normal populations from singly-truncated sample—*In English*

Rep. Statist. Appl. Res. (**JUSE**) (1959) **6**, 105-110 (2 references, 1 table, 5 figures)

In the estimation of the population mean μ and standard deviation σ from singly truncated normal samples, the maximum likelihood method is usually applied. This paper investigates the bias and precision of the maximum likelihood estimates $\hat{\mu}$ and $\hat{\sigma}$ of μ and σ (when σ is known, only $\hat{\mu}$ is concerned) by means of model experiments. n normal random numbers are taken using an appropriate table, μ and σ are estimated from r smaller values among n by the maximum likelihood method (when σ is known, only μ is estimated) and the deviations of estimates from true population values are calculated.

After these experiments are carried out repeatedly, say N times, the mean values of N deviations as the approximate biases and the standard deviations of $\hat{\mu}_i$ and/or $\hat{\sigma}_i$ as the approximate values of precisions $(\hat{\sigma}_i - \sigma)$ and/or $(\hat{\mu}_i - \mu) i = 1, 2, \ldots, N$ are calculated.

The author's conclusions deduced by these analyses of model experiments are as follows:

Case 1. μ and σ are both unknown.

(i) It seems that the distribution of $\hat{\mu}$ is exactly or approximately normal, but that of $\hat{\sigma}$ is not normal, at least when n and r are small.

(ii) The biases of the estimates $\hat{\mu}$ and $\hat{\sigma}$ are found to be of considerable magnitude.

(iii) Larger r for constant n and also larger n for constant r/n give estimates with smaller biases and standard deviations.

(iv) Cohen's asymptotic formulas [*Ann. Math. Statist.* (1950) **21**, 557-569] for the precision of estimates are found to give practically exact values when $n \geqq 30$.

Case. 2. σ is known and μ is unknown.

(i) The distribution of $\hat{\mu}$ seems to be exactly or approximately normal.

(ii) The bias is either zero or negligibly small.

(iii) The third and the fourth conclusions in Case 1 are valid also in this case. One example of obtaining closer estimates σ^* and μ^* than $\hat{\sigma}$ and $\hat{\mu}$ is given, in which the numerical results obtained by model experiments are used.

(M. Siotani)

2/55

KIMBALL, A. W. (Oak Ridge Nat. Lab., Tennessee) **4.3** (–.–)

Disease incidence estimation in populations subject to multiple causes of death—*In English*

Bull. Int. Statist. Inst. (1958) **36**, III 193-204 (8 references)

The analysis of mortality data presents special difficulties where one is interested in deaths due to one cause, owing to extraneous deaths or loss in the populations under study.

The work that has already been done, has been confined to human populations, with its attendant difficulties. Long term experiments with animals being both expensive and time consuming. However, following the need for information on whole body irradiation, well conducted experiments on mice have been made.

In this instance, the experimenter wishes to compare the instances of myeloid leukemia in male and female mice. One complication is that female mice tend to die from thymic leukemia before the risk of death from myeloid leukemia becomes appreciable.

The general problem considered is the estimation of disease incidences in a population and to account for the losses which have an origin other than that of the study.

It is assumed that the causes of death or loss operate independently but nothing is presumed about the form of distribution of survival times. The calculations of the estimates are simple. An illustrative example is given where there are three causes of death and one is interested in causes two and three. In the same example it is shown that if deaths from the first cause are ignored and incidences computed from remaining data, an unrealistic result is obtained. A formula is given which estimates the variance to a good approximation.

The exact form of the probability distributions of these estimations is not known, but preliminary results obtained by random sampling do not indicate an excessive systematic error. The formula given for the variance is shown to be satisfactory.

(M. J. Scott)

KELLERER, H. G. (Math. Inst., Univ. München)
Location parameters of n-dimensional distributions—*In German*
Math. Zeit. (1960) **73**, 197-218 (1 reference)

For the characterisation of the location of a n-dimensional distribution a location parameter is introduced by axioms. A continuous family of location parameters is constructed and their relation to well-known parameters is examined in particular by limit methods.

The n-dimensional random variables ξ_i are called convergent of degree $r>0$ to the random variable ξ, if for each vector t of the Euclidian space R_n the integrals from $-\infty$ to $+\infty$ over the absolute value of the difference of the distribution functions of $|\xi_i - t|^r$ and $|\xi - t|^r$ converge to zero. Convergence of any degree r implies convergence of the related distribution functions. The converse holds true, only if the random variables are bounded in probability.

A vector $L(\xi) \in R_n$ is called a r-continuous location parameter if L is r-continuous, in the sense of r-convergence defined above, and $L(T\xi) = TL(\xi)$ for each general similarity transformation T of R_n. Among other things it is proved that the expectation of a one-dimensional random variable is an l-continuous location parameter, and that it is the only one if L is required to be additive.

Furthermore, it is shown, that any function $\phi(\xi, t)$ is of the form $\phi(\xi, t) = \gamma |\xi - t|^r$ ($\gamma>0, r>1$), if the minimum point of $\mathfrak{E}[\phi(\xi, t)]$ is a location-parameter and

$$\frac{\mathfrak{E}(\phi(T\xi, t))}{\mathfrak{E}(\phi(\xi, T^{-1}t))}$$

is independent of t. The mean location $L_r(\xi)$ of the order $r>1$ is defined by the unique minimum point of $\mathfrak{E}[|\xi - t|^r]$. It is shown that $L_r(\xi)$ is a r-continuous location parameter and that $L_r(\xi)$ is the unique zero point of $\mathfrak{E}[|\xi - t|^{r-2}(\xi - t)]$.

As a certain measure of dispersion, the mean distance $k_r(\xi) = \{\mathfrak{E}[|\xi - L_r(\xi)|^r]\}^{1/r}$ of the order $r>1$ is introduced. An inequality analogous to the inequality of Tschebyscheff. can be proved. Finally, a mean difference $k'_r(\xi)$ of the order $r>1$ is defined by $\{\mathfrak{E}[|\xi_1 - \xi_2|^r]\}^{1/r}$, where ξ_1 and ξ_2 are independent and have identical distributions.

Differing from the usual definition, each minimum

KELLERER, H. G. (Math. Inst. Univ. München)
Location parameters of n-dimensional distributions—*In German*
Math. Zeit. (1960) **73**, 197-218 (1 reference)

point of $\mathfrak{E}[|\xi - t|]$ is called a median $M(\xi)$. If in a certain sense the range of ξ has a dimension greater than one the median $M(\xi)$ is uniquely determined. For one-dimensional random variables ξ_1 a principal median $H(\xi_1)$ is distinguished as the unique minimum point of $\mathfrak{E}[|\xi_1 - t| \log |\xi_1 - t|]$. On certain assumptions $L_r(\xi_1)$ converges to $H(\xi_1)$ for $r \to 1$. $H(\xi_1)$ satisfies the axioms of a location parameter except the continuity.

Let $V(\xi)$ be the set of all points x of R_n, for which each open set containing x has a probability >0 for ξ. If $V(\xi)$ is bounded, $L_r(\xi)$ converges for $r \to \infty$ to the unique minimum point of $\max |x - t|$ at which the maximum is to form over all $x \in V(\xi)$.

On certain assumptions $k_r(\xi)$ converges for $r \to \infty$ to $\inf \sup |x - t|$ at which the supremum is to form over all $x \in V(\xi)$ and the infimum is to form over all $t \in R_n$. On the same assumptions $k'_r(\xi)$ converges for $r \to \infty$ to $\sup |x_1 - x_2|$ at which the supremum is to form over all x_1 and x_2 of $V(\xi)$.

(W. Uhlmann)

RAJA RAO, B. (University of Poona) **4.8** (4.2)

A formula for the curvature of the likelihood surface of a sample drawn from a distribution admitting sufficient statistics—*In English*

Biometrika (1960) **47**, 203-207 (6 references)

A formula is derived for the curvature of the likelihood surface of a sample drawn from a distribution admitting sufficient statistics for m unknown parameters at the point represented by the maximum likelihood estimates of these parameters in terms of Fisher's information matrix. Use is made of a property of distributions which admit sufficient statistics; namely, that the expectation of the second derivative of the likelihood function with respect to any two parameters is identical to the same second derivative of the likelihood function when both are evaluated at the point represented by the maximum likelihood estimates of the parameters.

In the case of two parameters, the author shows that the first and second Gaussian curvatures of the likelihood surface are equal, respectively, to the negative of the trace and the determinant of the information matrix.

In general, for m parameters, the Riemann curvature invariant at the maximum likelihood point is equal to the negative of the sum of the products, two at a time, of the latent roots of the information matrix.

(J. G. Saw)

2/59

ROY, J. & CHAKRAVARTI, I. M. (Indian Statistical Institute, Calcutta) **4.9** (4.1)

Estimating the mean of a finite population—*In English*

Ann. Math. Statist. (1960) **31**, 392-398 (3 references)

The authors consider the problem of estimating the mean of a finite population consisting of N elementary units u_i. With each u_i is associated a real number

$$y_i (i = 1, 2, \ldots, N),$$

and the mean to be estimated is $\mu = N^{-1}\Sigma y_i$. The sampling procedures allowed are defined by a countable collection $\{U\}$ of finite or infinite sequences of elementary units with repetitions allowed. Each such sequence is called a sampling unit, and the probability that unit $U(x)$, x being an identifying integer for each member of this countable collection, is selected is $p(x)$. This formulation of the sampling process allows certain types of sequential sampling.

A linear estimator is defined as a random variable

$$T = t(X) = \sum_{i=1}^{N} y_i a_i(X),$$

where X is a random variable such that $\Pr(X = x) = p(x)$, $x = 1, 2, \ldots$, and the coefficients $a_i(x)$ are predetermined real numbers with the restriction that $a_i(x) = 0$ whenever the elementary unit u_i fails to appear in the sample whose serial number is x.

Among the class of linear unbiased estimators an admissible estimator T^* is found in Section 2 and applied to some specific sampling schemes in Section 4. In the special case of simple random sampling with replacement, this estimator corresponds to the mean of those y's which remain after all duplications with respect to elementary units are removed. Further, a complete class of linear unbiased estimators is defined.

In Section 3 a subset of the class of linear unbiased estimators, consisting of those that are linearly invariant, is defined. A counter-example is given to show that this subclass contains no best estimator.

(R. Hooke)

2/60

SAUNDERS, S. C. (Boeing Scientific Res. Labs., Seattle) **4.5** (4.7)
Sequential tolerance regions—*In English*
Ann. Math. Statist. (1960) **31**, 198-216 (13 references)

Wilks showed that the random proportion of a distribution falling between two order statistics (the "coverage") does not depend on this distribution and has itself an easily computed distribution. This formed the basis of the tolerance intervals which he introduced for a one-dimensional real random variable. Jiřina [*Czech. Mat. Z.* (1952) **77**, 211-232] made the first attempt to study sequential distribution-free tolerance limits. As in Wilks' case, the tolerance limits are two prescribed order statistics but the sample size is not fixed in advance but rather sampling terminates when these two order statistics have remained unchanged for k observations.

In the present paper a more general sequential procedure is defined; a tolerance region based on the order statistics is defined for each finite sample. As in the Jiřina procedure, sampling terminates when the region remains unchanged for a prescribed number of observations. The random coverage of the region so determined is distribution free. The author establishes some relationships between the distribution of the coverage and the generating function of the random sample size, which permits the determination of one in terms of the other. Formulae are obtained for the expected sample size of the Jiřina procedure.

Comparisons are made between fixed and sequential sample procedures. For example, it is shown that for one-sided tolerance limits no sequential procedure is as good as Wilks' fixed sample procedure in the sense that if the expected sample sizes are the same the coverages of the Wilks' procedure is greater than the coverages of the sequential procedures. For two-sided tolerance limits there is no uniformly best procedure.

It should be pointed out that while the usual formulation of the tolerance interval problem is for real random variables the formulation in this paper is for random variables defined on any space with a linear ordering and a probability measure that assigns probability zero to each equivalence class induced by the ordering.

(D. G. Chapman)

2/61

SIOTANI, M. (Institute of Statistical Mathematics, Tokyo) **4.8** (7.7)
Notes on multivariate confidence bounds—*In English*
Ann. Inst. Statist. Math., Tokyo (1960) **11**, 167-182 (4 references, 6 tables)

This paper treats the problems of setting the simultaneous confidence bounds for comparisons among several normal population means and the evaluation of the upper 100α percentage points of the extreme statistics appearing in the construction of the bounds. Suppose that we have k p-variate normal populations with mean vectors $\mu_i (i = 1, \ldots, k)$ and the common covariance matrix Λ, that is, $N(\mu_i, \Lambda)$, $i = 1, \ldots, k$; the following three problems are discussed:

(i) to give simultaneous confidence bounds on a specific set of independent comparisons among μ_i.
(ii) when one of k populations, for example, the kth population $N(\mu_k, \Lambda)$, corresponds to the standard experiment, to set simultaneous confidence bounds on $a'(\mu_i - \mu_k)$ for all non-null p-dimensional scalar vectors a's and all $i = 1, 2, \ldots, k-1$, and
(iii) to give simultaneous confidence bounds on the contrasts, $a'(\mu_i - \mu_j)$ for all non-null a's and all $i, j = 1, 2, \ldots, k (i \neq j)$.

In these problems, the maximum of a certain number of Hotelling's T^2 statistics or distances (in a different form for each case) appears. By using the method given by the author in his previous paper [*Ann. Inst. Statist. Math.* (1959) **10**, 183-208: and abstracted in this journal No. 417, **5.5**] the approximate upper 100α percentage points of these maximal statistics are evaluated and tables for $\alpha = 0{\cdot}05$, and $0{\cdot}01$ are given for cases (i) and (ii). Since the method of evaluation does not have the analytical bounds on errors, the numerical examination is carried out for the accuracy of the approximation.

(M. Siotani)

2/62

STUART, A. (London School of Economics)
The measurement of estimation and test efficiency—*In English*
Bull. Int. Statist. Inst. (1958) **36**, III 79-86 (18 references)

4.1 (5.1)

Methods of measuring test efficiency have taken longer to develop, and are intrinsically more complicated, than those for measuring the efficiency of estimators. Consequently they are less well known.

This expository paper deals with the development of both topics, commencing with Fisher's celebrated concept of asymptotic efficiency for normally distributed estimators which differ in respect of the dispersion of their distributions about the true value. The use of the inverse variance ratio in this situation is shown to be appropriate only when the variance of the most efficient estimator is of order $1/n$. The implications are considered of variances of other orders, including the case where all inefficient estimators have zero efficiency. For this method, the choice of variance as the measure of dispersion is merely one of convenience and in theory it does not matter which measure of dispersion is used.

The implications of extending the use of the inverse variance ratio to the sample variances of small samples are next considered, and this method is contrasted with the use of the Intrinsic Accuracy Ratio which failed to be adopted for general use.

Turning to test efficiencies, the author explains how the need for measuring the relative efficiencies of alternative tests was precipitated in the decade following 1935 by the rapid development of distribution-free tests. This led to the idea of power efficiency, but the need for a more concise description of test performance was soon apparent. The result was the criterion of Asymptotic Relative Efficiency, the usefulness and limitations of which are discussed in various contexts. It is explained that asymptotic relative efficiency is a measure only of *local* power (i.e. in the region of the null hypothesis). The need for generalisation led to the Cox-Stuart determination of the maximum difference between the asymptotic power functions of comparable tests and of the values of these functions at the point of maximum difference.

Finally, it is emphasised that asymptotic normality is not essential to the use of asymptotic relative efficiency as a measure of test efficiency, provided that the power functions being compared are of the same form. Its use with the non-central chi-square distribution is briefly discussed.

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(D. J. Page)

ANDERSON, T. W. (Columbia University, New York) **5.7** (5.1)

A modification of the sequential probability ratio test to reduce the sample size—*In English*

Ann. Math. Statist. (1960) **31**, 165-197 (9 references, 2 tables)

Let X_1, X_2 X_3, ... be a sequence of independently distributed normal random variables with mean μ and known variance σ^2. The sequential probability ratio test of the hypothesis $H_0 : \mu = -\mu^*$ against the alternative $H_1 : \mu = \mu^*$ is known to have certain optimum properties, viz. among all tests with the same Type I and Type II errors the expected sample size when $\mu = -\mu^*$ or $\mu = +\mu^*$ is minimised by the sequential probability ratio test. However, if in fact μ lies near 0, that is near the midpoint of the hypothesis and alternative, then the expected sample size may be unreasonably large.

In the sequential probability ratio test observations are continued as long as

$$b < \sum_{i=1}^{m} X_i < a$$

for some a, b. This test can be described graphically by plotting successive points

$$(1, X_1)\,(2, X_1+X_2)\,(3, X_1+X_2+X_3)\ldots$$

in the plane $(m, y = \Sigma X_i)$. As soon as a point falls on or above $y = a$, H_0 is rejected while if it falls on or below $y = b$, H_1 is rejected. Otherwise sampling continues.

In this paper the parallel straight lines $y = a$ and $y = b$ are replaced by straight lines $y = c_1+dm$; $y = c_2+dm$ with possibly truncation at N: that is a line $m = N$. To calculate the probabilities and expected values that are of interest for this modified sequential test the sequence ΣX_i is replaced by the Wiener stochastic process $X(t)$. For this process, which then represents an approximation to the original problem, the various probabilities of error and expected time to decision are found. Unfortunately, they are extremely complicated functions of μ and of the constants defining the procedure. Two cases have been investigated computationally and some numerical results are given.

(D. G. Chapman)

2/65

CHAKRAVARTI, I. M. & RAO, C. R. (Indian Statistical Institute, Calcutta) **5.2** (11.1)

Tables for some small sample tests of significance for Poisson distributions and 2×3 contingency tables—*In English*

Sankhyā (1959) **21**, 315-326 (11 references, 8 tables)

In an earlier paper, see Rao & Chakravarti [*Biometrics* (1956) **12**, 264] provided exact tests for (i) homogeneity of samples and goodness of fit for Poisson and truncated Poisson populations and (ii) deviation in the "zero frequency" for Poisson and Binomial populations. The same paper also contained tables of critical values, at the level of significance, 0·05, for the variance and likelihood tests for homogeneity, goodness-of-fit and deviation in the zero frequency for samples from a Poisson population. In the previous paper tables were given for combinations of $T = 1(1)10$ and $f = 3(1)10(10)100$, where T and f denote the total of the observations from a Poisson population and for the sample size. In the present paper the tables are extended to $T = 11$ and 12 for the same values of f.

This paper also contains the table of critical values of the statistics given by the sum of squares of observations from a truncated Poisson population for $f = 3(1)9$ and $T = 8(1)12$.

The authors also consider g sets of s binomial trials and the problem of testing the equality of binomial probabilities. If $x_1 \ldots x_g$ denote number of successes in the different sets, the variance test and likelihood test for homogeneity are provided by the statistics Σx_i^2 and $\Sigma x_i \log x_i + \Sigma(s-x_i)\log_e(s-x_i)$ respectively. The authors give the critical values of these statistics for $g = 3$, $s = 3(1)10$, $T = 3(1) = 6$. Tables of $n \log_e n$ for $n = 1(1)100$ are given to facilitate the computation of the likelihood test statistic.

Examples are given illustrating the use of tables.

(Y. R. Rau)

DIAMOND, E. L., MITRA, S. K. & ROY, S. N. (University of North Carolina, Chapel Hill) **5.1** (6.9)
Asymptotic power and asymptotic independence in the statistical analysis of categorical data—*In English*
Bull. Int. Statist. Inst. (1960) **37**, III 309-329 (8 references)

This is a brief report on some aspects of the work done by Mitra (with Roy) and Diamond (reported in North Carolina Institute of Statistics Mimeograph Series No. 142 and 196) following the paper by Roy & Mitra [*Biometrika* (1956) **43**, 361-376]. Given a product-multinomial distribution of the form:

$$\prod_j \left[(n_{oj}! / \prod_i n_{ij}!) \prod_i p_{ij}^{n_{ij}} \right]$$

(where $\sum_i n_{ij} = n_{oj}$ is fixed and $\sum_i p_{ij} = p_{oj} = 1$

and both i and j may be multiple subscripts), a test is derived for the composite $H_0 : p_{ij} = p_{ij}(\theta_1, \ldots, \theta_t)$ subject to the constraints on θ's : $f_k(\theta_1, \ldots, \theta_t)$ $0 (k = 1, 2, \ldots, u < t)$ where t is the number of cells minus the number of separate multinomial distributions. This test is based on a statistic which, in the limit as $n \to \infty$ subject to n_{oj}/n's being held fixed, has the χ^2-distribution with degrees of freedom equal to the number of independent parameters in the model minus the number of independent parameters in the hypothesis.

Asymptotic power (in the sense of Pitman) of the test of this hypothesis is obtained, in the limit as $n \to \infty$ subject to n_{oj}/n's being held fixed, against the sequence of simple alternatives $Hn : p_{ijn} = p_{ij}(\theta_1^0, \ldots, \theta_t^0) + \delta_{ij}/\sqrt{n}$ where δ_{ij}'s are given subject to $\sum_i \delta_{ij} = \delta_{oj} = 0$ and $\theta_1^0, \ldots, \theta_t^0$ is a specific point on the H_0 surface subject to the side constraints: some invariance properties are derived. Next, the first part of the foregoing hypothesis is moved into the model, the second part alone being retained in the hypothesis, and a test is obtained for this hypothesis and also its asymptotic power against the corresponding sequence of simple alternatives suitably defined.

The asymptotic independence of two sets is defined and a necessary and sufficient condition is obtained for the asymptotic independence of the two tests for two such hypotheses. Finally, the general theorems are specialised to be able to deal with various problems of statistical interest, which are analogous to those in " normal " analysis of variance and multivariate analysis.

(E. L. Diamond)

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DUNNET, C. W. (American Cyanamid Co., Pearl River, N.Y. and Univ. of Aberdeen) **5.8** (1.8)
On selecting the largest of k normal population means—*In English*
J. R. Statist. Soc. B (1960) **22**, 1-30 (31 references, 2 tables, 4 figures)

Consider a k-variate normal distribution with unknown means and with equal known variances and equal known covariances; the problem is to select the variate with the largest mean using a single-sample procedure of size n, and to determine the required value of n. Previous work is very fully reviewed. New results are given assuming the means have a joint distribution which is *a priori* normal, with means known up to an additive constant, and with equal known variances and equal known covariances.

Various formulations are considered. Results are given for the probability of correct selection; for the probability taken conditionally on the largest mean exceeding the others by at least a specified amount; and for the probability of the selected mean being within a specified amount of the largest, this being equivalent to considering a certain zero-one loss function. The results are in terms of $(2k-2)$-dimensional normal integrals that can be evaluated only if $k = 2$; comparisons with Bechhofer's procedure are made for this case.

A loss function is considered which is linear in the difference between the largest mean and that selected, with linear experimentation costs. The choice of k and n is discussed, and the relation with the plant-selection problem of maximising the " genetic advance " is pointed out. These results involve $(k-1)$-dimensional normal integrals; to find the minimax and " largest admissible " values of n only single integrals are needed.

Research is needed on:

(i) multiple characteristics;
(ii) non-normality;
(iii) sequential procedures;
(iv) comparison with a standard population.

In an appendix the expected value of the largest of the k normal variates with equal correlations and variances —but different means—is obtained in terms of single integrals.

(C. L. Mallows)

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DUNNETT, C. W. (Report of discussion of a paper by)
On selecting the length of k normal population means—*In English*
J. R. Statist. Soc. B (1960) **22**, 30-40

5.8 (1.8)

The contributors to the discussion were Armitage, Johnson, Nelder, Miss Rita Maurice, Lindley, Good, Cox, Barnard and Finney: the author replied.

Armitage pointed out the considerable variation between comparative experiments; they range from scientific experiments having few direct consequences to selection procedures. Also we should distinguish between the author's problem of selecting a fixed number or proportion of the given items, and that of problems where the absolute quality is relevant. In practice it may be difficult to formulate the prior knowledge and utilities required for the Bayesian solution; there is opportunity for a sequential approach in which the selection rules are generally modified. There is a connection between the present situation and Crossman's theory of the effect of practice in reducing the time taken to perform a task.

Johnson showed that in the case of two normal populations, the concept of largest admissible sample size could be extended to sequential procedures. He questioned the suitability in practice of assuming that the prior distributions are normal. If this is accepted then in principle, a sequential procedure can be constructed based on the posterior probabilities.

Nelder discussed the distribution between problems of internal economy, where total resources are given, and of external economy, where they are at choice. He asked for asymptotic expansions of the multivariate integrals when k is large.

Miss Maurice had calculated an envelope-risk-function for sequential procedures in the two-population case. She discussed the situation of differing population variances and/or costs of experimentation. The maximum principle can be used, and the reduction in the minimax expected loss can be calculated (for two populations).

Lindley said that a " prior distribution with unknown means " was a contradiction in terms; also utilities cannot always be described in economic terms. He had applied the technique of dynamic programming to the two-population sequential selection problem and had obtained a simple differential equation; however, the

DUNNETT, C. W. (Report of discussion of a paper by)
On selecting the length of k normal population means—*In English*
J. R. Statist. Soc. B (1960) **22**, 30-40

5.8 (1.8)
continued

boundary is unknown and the method has not yet yielded a definite answer. Good pleaded for more work on sequential procedures, and discussed the theory of an American tournament (i.e. a round robin). Cox pointed out that in some cases the number of populations selected should be allowed to depend on the observations. If a two-point prior distribution was assumed, plausible sequential procedures could be specified.

Barnard described how prior distributions might be obtained by preliminary surveys, and how the information contained in them could be presented visually. Finney remarked that in the discussion nobody had mentioned a significance test; at the practical level of consultation work, this classical notion is often irrelevant.

(C. L. Mallows)

EEDEN, Constance van (Mathematical Centre, Amsterdam) **5.4** (5.6)

A class of tests for the hypothesis that K parameters $\theta_1, \ldots, \theta_k$ satisfy the inequalities $\theta_1 \leqq \ldots \leqq \theta_k$—*In English*

Bull. Int. Statist. Inst. (1960) **37**, III 331-338 (7 references)

In this paper the following problem is treated: let $x_1, \ldots, x_k$ be k independent random variables and let θ_i denote an unknown parameter of the distribution of x_i. Further, for each $i = 1, \ldots, k$, let n_i independent observations of x_i be given. A test is described for the hypothesis H_0 that $\theta_1, \ldots, \theta_k$ are monotonically non-decreasing against the alternative hypothesis that it is not so.

The test is performed as follows. Let, for $i = 1, \ldots, k-1$, T_i denote a test for the hypothesis $\theta_i \leqq \theta_{i+1}$ against the alternative hypothesis $\theta_i > \theta_{i+1}$ with test-statistic t_i and critical region Z_i of size α_i. The hypothesis H_0 is rejected if and only if a value of i exists with $t_i \in Z_i$.

This test for H_0 possesses the following properties. Let α_0 denote the size of the critical region and let $\alpha \overset{\text{DEF}}{=} \sum_{i=1}^{k-1} \alpha_i$, then $\alpha_0 \leqq \alpha$. Further, if the tests T_i satisfy certain conditions, the probability of rejecting H_0 if $\theta_1 = \ldots = \theta_k$ at least equals $\alpha - \frac{1}{2}\alpha^2$. The asymptotic properties of the test for H_0 follow from those of the tests T_i. If certain conditions are satisfied concerning the asymptotic power of the tests T_i, the probability of rejecting H_0 if H_0 is not true tends to 1 if all n_i tend to infinity and the probability of rejecting H_0 if $\theta_1 < \ldots < \theta_k$ tends to zero if all n_i tend to infinity.

Situations where the above-mentioned conditions are satisfied are, for example, the cases where T_i is the likelihood ratio test for

(i) rectangular distributions between 0 and θ_i,
(ii) exponential distributions with mean θ_i^{-1},
(iii) normal distributions with variance θ_i,
(iv) normal distributions with mean θ_i and known variance σ_i^2.

The author also considers an analogous distribution-free test based on Wilcoxon's two-sample test. In this paper no proofs are given; these may be found in the author's thesis, [*Testing and estimating ordered parameters of probability distributions*, Amsterdam, 1958].

(Constance van Eeden)

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HAGA, T. (Sanyo Pulp Co., Ltd., Japan) **5.6** (3.8)

A two-sample rank test on location—*In English*

Ann. Inst. Statist. Math., Tokyo (1960) **11**, 211-219 (3 references, 2 tables, 5 figures)

Rosenbaum's two-sample rank tests [*Ann. Math. Statist.* (1953) **24**, 663-668 and (1954) **25**, 146-150] on the homogeneity of locations and dispersions are based on the so-called number of exceedances. The power function of the test for location is strongly affected by the difference of population dispersions, and the test for dispersion by the difference of population locations.

Kamat [*Biometrika* (1956) **43**, 377-387] modified Rosenbaum's test for dispersion, so that the above-mentioned change of power be small. His statistic is (range of ranks of y's) − (range of ranks of x's) + m ($m \geqq n$) in a pooled arrangement in order of magnitude of x's and y's, where m and n are sample sizes of x and y, respectively.

Analogously, if we use the statistic {(the number of x larger than all y or y smaller than all x) − (the number of y larger than all x or x smaller than all y)}, we obtain the test for slippage rather less affected by the difference of dispersion.

The distribution of the statistic is discussed, and the tables for both the new and Kamat's tests are presented with figures showing the power.

(M. Sibuya)

2/72

ELTEREN, P. van (Catholic University, Nijmegen)

On the combination of independent two-sample tests of Wilcoxon—*In English*

Bull. Int. Statist. Inst. (1960) **37**, III 351-361 (12 references)

In this paper a class of tests is considered for which the test-statistics are linear combinations of statistics of independent two-sample Wilcoxon tests (also named after Mann & Whitney). The tests of this class can be used for testing the hypothesis that both samples of each pair, that is of each individual Wilcoxon test, have been taken from the same population—which may be different for different pairs. Such a combination-statistic is approximately normally distributed if either the number k of combined tests or the sample sizes (m_i and n_i, $i = 1, \ldots, k$) of each individual test are not too small. This is derived from the central limit theorem and related theorems. The mean and variance of the statistic under the null hypothesis and thus the approximate critical values are easily obtained.

Two special combinations are particularly investigated. The coefficients of these combinations are respectively:

(i) proportional to the inverse of the product of the sample sizes of each test,

(ii) proportional to the inverse of the sum of the sample sizes plus unity.

It is shown that the combination of the first type generates a so-called " design-free " test, that is to say a test with a region of consistency independent of the ratios of the sample sizes of the same and of different pairs.

The combination of the second type gives a test which is locally the best of all the combinations considered here against the following special alternatives. Let x_i and y_i be the random variables sampled in the ith pair with distribution functions $F_i(x)$ and $G_i(x)$ respectively, and let the number k of pairs tend to infinity. Let $F_i(x)$ and $G_i(x)$ depend on k in such a way that

$$\sqrt{k}\{G_i(x)-F_i(x)\}$$

is bounded for $k\to\infty$. Let b_i be defined by

$$b_i = 2P[x_i>y_i]-1 = 2\int G_i(x)dF_i(x)-1.$$

It then follows that the combination of the second type has (asymptotically, $k\to\infty$) the largest power against all alternatives for which the ratio of each couple of b's tends to unity. A similar result is derived for the case

ELTEREN, P. van (Catholic University, Nijmegen)

On the combination of independent two-sample tests of Wilcoxon—*In English*

Bull. Int. Statist. Inst. (1960) **37**, III 351-361 (12 references)

that k is constant, but that m_i and n_i ($i = 1, \ldots, k$) are proportional to a number N that tends to infinity.

In the last section of this paper the author makes comparisons for different alternatives between the efficiencies of both the combinations which he has discussed. He draws the following conclusions:

(i) If large differences between the b_i are possible the design-free combination is preferable as in the case of large differences the locally best test is not optimal and its consistency conditions depend strongly on the sample sizes.

(ii) If it is reasonable to assume that the b_i have values close to zero the locally best test may be preferred because of its larger efficiency.

(P. van Elteren)

HEMELRIJK, J. (Mathematical Centre, Amsterdam)

5.6 (4.6)

Distribution-free tests against trend and maximum likelihood estimates of ordered parameters—*In English*

Bull. Int. Statist. Inst. (1958) **36**, III 15-25 (18 references)

This paper is concerned with a review of a number of publications on the two subjects mentioned in the title. In particular the author is concerned with bringing under one heading the work done in allied fields by different people in different parts of the world.

The problem considered in the first part of the paper is that of testing for trends, i.e. given k random variables with a varying number of observations on each, the hypothesis that they all have the same probability distribution is tested against the alternative of an upward or downward trend. This problem can be solved directly by the use of Kendall's rank correlation method. For the case of single readings on the variates this has been done by Mann [*Econometrica* (1945) **13**, 245-259], and Terpstra [*Indag. Math.* (1952) **14**, 327-333, (1955) **17**, 690-696 and (1956) **18**, 59-66] has dealt with the case of samples of all sizes. There is a similarity between Kendall's test and Wilcoxon's test [*Biometrics* (1945) **1**, 80-82]. However the consistency of the test statistic depends on the sample size, and a small modification is required. The theory of the modified test is worked out in Terpstra's later papers. In this present paper the author gives, without proof, the conditions for consistency. The second problem considered in this part of the paper is the special case of the random variates being binomially distributed. This problem has been considered by the author and Constance van Eeden [*Indag. Math.* (1955) **17**, 191-198 and 301-308]. The test statistic is asymptotically normally distributed and the conditions for this together with those for consistency are given in the paper.

The second part of the paper is concerned with the estimations of ordered parameters. The problem is as follows: there are available k observations associated with k distributions: it is required to estimate the k distribution parameters, given that; the parameters are, firstly, confined to given intervals and secondly, partially or completely ordered.

This problem has been solved by H. D. Brunk [*Ann. Math. Statist.* (1955) **26**, 607-616] under certain conditions. Brunk gives explicit formula for the parameter

HEMELRIJK, J. (Mathematical Centre, Amsterdam)

5.6 (4.6) *continued*

Distribution-free tests against trend and maximum likelihood estimates of ordered parameters—*In English*

Bull. Int. Statist. Inst. (1958) **36**, III 15-25 (18 references)

estimates, however in another paper [*Indag. Math.* (1957) **19**, 506-512] Miss van Eeden has developed theorems and lemmas which although they do not contain the parameter estimates explicitly, lead to simpler computations. An outline of Miss van Eeden's method is given in this paper.

A special case of the above problem is that where the parameters are probabilities. This case has no special features and was solved by Miriam Ayer, H. D. Brunk *et al.* [*Ann. Math. Statist.* (1955) **36**, 641-647] and by Miss van Eeden in a further paper [*Indag. Math.* (1956) **18**, 444-455].

(A. N. James)

KITAGAWA, T. (Kyushu University, Fukuoka, Japan) **5.1** (4.2)

Successive process of statistical inference associated with an additive family of sufficient statistics—*In English*

Bull. Int. Statist. Inst. (1958) **36**, III 26-36 (15 references)

In this paper the author proposed the concept of the "additive family of sufficient statistics" together with relative statistics associated with the particular family under certain restricted conditions. The paper shows how the concepts are useful for developing aspects of the two-sample and k-sample problems and which appear to be of basic importance for successive processes of statistical inference—as already developed by the author during the period from 1950: references are given in the paper. The concept of the additive family of sufficient statistics may form an acceptable approach to the understanding of fiducial arguments.

The second section of this paper defines the conditions for which independent sufficient statistics of a parameter may be an additive family of such statistics with respect to that parameter. This section also does the same for the relative statistic associated with the particular family. The following section sets out some characterisations of the additive family and it may be noted that the expression for the necessary and sufficient conditions can be reduced to a normal form by the use of an appropriate transformation.

The fourth section uses the decomposition of two independent additive statistics for the two-sample problem. Three interpretations of a simple formula are given—the prediction, test of significance and fiducial argument—and the author prefers the second. The fifth section discusses the k-sample problem and associated limiting procedure and the last two sections deal with the problem of estimation after a preliminary test of significance and application of Wald's sequential probability-ratio test to an additive family of sufficient statistics.

(W. R. Buckland)

LANCASTER, H. O. (University of Sydney) **5.1** (6.9)

On tests of independence in several dimensions—*In English*

J. Aust. Math. Soc. (1960) **1**, 241-254 (34 references, 2 tables)

The author considers tests of independence for random variables X, Y, Z having the joint distribution function $F(x, y, z)$, and marginal distributions $G(x)$, $H(y)$, $K(z)$, subject to the condition that

$$\Phi^2+1 = \int dF^2\{dG\, dH\, dK\}^{-1} < \infty.$$

The distribution may be expanded in a series

$$dF = \Big\{1+\sum_{jk}\rho_{ojk}y^{(j)}z^{(k)}+\sum_{ik}\rho_{iok}x^{(i)}z^{(k)}+\sum_{ij}\rho_{ijo}x^{(i)}y^{(j)} + \sum_{ijk}\rho_{ijk}x^{(i)}y^{(j)}z^{(k)}\Big\}\, dG\, dH\, dK$$

where $\{x^{(i)}\}$, $\{y^{(j)}\}$, $\{z^{(k)}\}$ are complete orthonormal sets defined on the distributions G, H, K respectively, the ρ_{ijo}, ρ_{iok}, ρ_{ojk} being first order interactions and the ρ_{ijk} second order interactions.

It is shown that $\Phi^2 = \Phi^2_{xy}+\Phi^2_{yz}+\Phi^2_{xz}+\Phi^2_{xyz}$ where $\Phi^2_{xy} = \Sigma\rho^2_{ijo}$, $\Phi^2_{yz} = \Sigma\rho^2_{ojk}$, $\Phi^2_{xz} = \Sigma\rho^2_{iok}$, $\Phi^2_{xyz} = \Sigma\rho^2_{ijk}$, and further that $\Phi^2 = 0$ is a necessary and sufficient condition for complete independence of X, Y, Z.

The pattern of dependence or independence of the random variables X, Y, Z in such distributions can be classed into eight types according to whether Φ^2_{xyz} is equal to zero or not, and the number of the Φ^2_{xy}, Φ^2_{yz}, Φ^2_{xz} which are zero. An independence table for these is provided, and the partitioned χ^2-method of testing independence is considered.

Bartlett's treatment of the two by two by two contingency table is discussed, and applied to a random sampling experiment. The results are shown to be comparable with those obtained by the partition of χ^2, but the partitioning methods are more general.

(J. Gani)

MALÝ, V. (Charles University, Prague) **5.7** (4.7)

Sequential problems with multiple decisions and sequential estimation—*In German*

Biom. Zeit. (1960) **2**, 45-64 (5 references, 7 figures)

This paper is the first part of a series of papers on sequential tests and medical trials. It contains the theory and applications to binomial distribution. The author modifies the method by de Boer for testing three hypotheses concerning the mean of a normal distribution. Let n be the number of independent observations (trials), r the number of successes, $s = n-r$ the number of failures, and p the unknown probability of success. The interval $\langle 0; 1\rangle$ is divided in k parts by the points $x_1 < x_2 < \ldots < x_{k-1}$; the extreme points are $x_0 = 0$ and $x_k = 1$. A quantity $d > 0$ is chosen such that

$$x_i + d < x_{i+1} - d \quad (i = 1, 2, \ldots, k-2)$$

and

$$0 < x_1 - d; \qquad x_{k-1} + d < 1.$$

The hypotheses $H_i : x_{i-1} \leqq p_i < x_i$ are tested by $k-1$ independent sequential procedures with two decisions. The test T_i proceeds as follows: the hypothesis H_{i-}: $p = x_i - d$ tested against $H_{i+} : p = x_i + d$. The result is either R_{i+} (accepting H_{i+}) or R_{i-} (accepting H_{i-}). The probabilities of the $2(k-1)$ possible wrong decisions are supposed to be α. There exists one point x_h, such that for $i \geqq h$ the hypotheses H_{i-}, and for $i < h$ the hypotheses H_{i+} are accepted. The series of the results is then represented by:

$$R_{1+}, R_{2+}, \ldots, R_{(h-1)+}, R_{h-}, R_{(h+1)-}, \ldots, R_{(k-1)-}.$$

The author demonstrates his ideas for a less complicated system. The error of the first kind and the error of the second kind are both equal to α. It is useful to apply graphical methods. In the sequential plan every sample leaves a " trace " and all traces together form a " trace-line ". As an example the graph of the sequential plan for $k = 10$, $x_i = i/10$, $d = 0{\cdot}04$, $\alpha = 0{\cdot}05$ is given. The importance of the limiting lines $r - s = \pm n$ is explained and the meaning of the quantity d. The expected size of the sample $\mathscr{E}(n)$ depends on the area between the two straight lines. As $\mathscr{E}(n)$ shall be as small as possible, the quantity d is to be chosen as big as possible:

$$\max d = \tfrac{1}{2} \min (x_{i+1} - x_i).$$

It is possible to define the true level of significance of the test.

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MALÝ, V. (Charles University, Prague) **5.7** (4.7) *continued*

Sequential problems with multiple decisions and sequential estimation—*In German*

Biom. Zeit. (1960) **2**, 45-64 (5 references, 7 figures)

In order to study the binomial distribution, the author uses three hypotheses:

$$\left.\begin{matrix} H_+ \\ H_p \\ H_- \end{matrix}\right\} \text{ where the unknown parameter is } \begin{cases} p+d \\ p \\ p-d \end{cases}$$

Results are given concerning the precision for estimating p by means of the sequential method.

(G. Reissig)

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PFANZAGL, J. (Inst. Statist., University of Vienna) **5.6** (5.3)

On locally optimum rank tests—*In German*

Metrika (1960) **3**, 143-150 (6 references)

Let N functions $h_i(u, \theta)$ be defined for $0 \leqq u \leqq 1, 0 \leqq \theta \leqq \theta_0$, such that $h_i(0, \theta) = 0$, $h_i(1, \theta) = 1$, $h_i(u, 0) = u$, $h_i(u, \theta)$ being strictly monotonic as a function of u. The mixed second derivative of $h_i(u, \theta)$ to u and θ at the point $\theta = 0$ is denoted by $h_i(u)$. Let $X_1 \ldots, X_N$ be independent random variables, X_i distributed with $h_i(F(x), \theta)$, where $F(x)$ is a continuous distribution function. Let $U^{(r)}$ be the rth, ordered by magnitude, of N independent random variables which are uniformly distributed in [0, 1]. By using the lemma of Hoeffding and the fundamental lemma of Neyman and Pearson, it is shown that the test-statistic $\Sigma \mathscr{E}[h_i^*(U^{(r,i)})]$ yields a rank test for the hypothesis $\theta = 0$, against the alternative $\theta > 0$, which is locally optimum at $\theta = 0$.

By using the results of an earlier paper (abstracted in this journal No. 235, **5.2**) the author shows that the absolute value of this test-statistic yields a locally optimum test- and classification-process for the hypothesis $\theta = 0$ against the alternatives $F_i(x) = h_i[F(x), \theta]$ and $F_i(x) = h_i^{-1}[F(x), \theta]$. Optimality in this sense means that the probability of a right decision is maximised subject to a given level for testing the hypothesis assuming equal *a priori* probability for both alternatives.

These results are applied to the two sample problem, that is, to the case $h_i(u, \theta) = h(u, \theta)$ for $i = 1, \ldots, m$ and $h_i(u, \theta) = u$ for $i = m+1, \ldots, N$. It follows from the above that, for each locally optimum one-sided test, a locally optimum test and classification process is obtained by using the absolute value of the test-statistic. Therefore, the optimality of the one-sided Wilcoxon (Mann-Whitney) test for mixture-alternatives as shown by Lehmann immediately implies the optimality of the two-sided Wilcoxon test regarded as a combined test- and classification-process. In the same way, with regard to translation-alternatives, is the optimality of the two-sided analogues of the tests of Fisher and Yates (locally optimum against translation alternatives for the normal distribution). Again, the Wilcoxon test (locally optimum against translation alternatives for the logistic distribution) is obtained.

Finally, tests are mentioned which are sensitive for trend alternatives. For this purpose trends are defined by $h_i(u, \theta) = h(u, i\theta)$, which is for $\theta \to 0$ equivalent to $h_i(u, \theta) = h(h_{i-1}(u, \theta))$ with $h_0(u, \theta) = u$.

(W. Uhlmann)

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PRATT, J. W. (Harvard University) **5.6** (4.6)

Remarks on zeros and ties in the Wilcoxon signed rank procedures—*In English*

J. Amer. Statist. Ass. (1959) **54**, 655-667 (10 references, 1 figure)

The Wilcoxon signed rank procedures are discussed and some inconsistencies resulting from the reduced sample procedure for handling zeros are pointed out. It is shown, by examples, that the reduced sample procedure can lead to judging a particular sample significantly positive, but then if these sample values are changed in a certain way to make the sample actually more positive, the new sample would be judged not significantly positive. Somewhat analogous anomalies occur with confidence interval estimation of the centre of symmetry μ, for example the confidence interval might have $+1$ as a lower end point to an interval and also include the point 0, thus excluding values between zero and unity.

Three equivalent requirements are given which a procedure in the case of zeros should satisfy to avoid the difficulties previously noted. A procedure is then given which satisfies the necessary requirements and an example is given comparing the conclusions drawn from the reduced sample procedure and the newly proposed procedure.

If some observations are tied in absolute value it is usually suggested that their ranks be averaged before attaching signs (average rank procedure). It is then shown that this average rank procedure likewise leads to some anomalies and does not satisfy intuitive requirements. A modification of the average rank procedure is discussed as to its comparative merits and demerits. Finally, several additional methods suggested by various people for handling ties are discussed.

(H. Larson)

SALVEMINI, T. (University of Rome) **5.6** (2.6)

Distribution of the range when sampling sets of numbers from a universe of equidistributed numbers—*In French*

Bull. Int. Statist. Inst. (1958) **36**, III 69-77 (2 references)

The author derives the distribution of the range when sampling sets of numbers from a universe of consecutive integer numbers, each of which are repeated a constant number of times. He studies the limit of this distribution, when the universe extends towards infinity, and compares this result with the one applicable to a rectangular distribution. He shows, in a numerical case, the remarkable differences in results, under different hypotheses concerning the distribution of the universe or the sampling scheme.

(T. Salvemini)

SINGH, B. N. (Indian Standards Institutions, New Delhi) **5.3** (5.6)

On the application of the statistics W and T for testing two samples—*In English*

J. Indian Soc. Agric. Statist. (1958) **10**, 107-130 (8 references, 6 tables)

Iyer & Singh [*J. Indian Soc. Agric. Statist.* (1955) **7**, 127] introduced the statistics W and T for testing the randomness of a given sequence of observations and the homogeneity of two or more samples. For a sequence of n observations taken from a continuous or discontinuous population, the statistics W and T are obtained as follows: form $(n-r+1)$ blocks of r consecutive observations each; consider the differences, taken in the same sense (i.e. the right element should be subtracted from the left or vice versa), between all possible pairs in each block and assign scores of 1 for a positive difference and -1 for a negative difference. The total number of positive (or negative) differences for all the blocks is W_r. The number of positive differences for all pairs separated by $r-2$ observations, each difference being taken once only, is represented by T_r.

In case the sequence consists of n elements of two kinds, A and B, W_r will be the total number of AB pairs (or BA pairs) in the $(n-r+1)$ blocks. T_r is obtained similarly. It is known that these statistics provide consistent tests and are asymptotically normally distributed.

In this paper, the author makes a comparative study of the power functions of the statistics W, T and the Mann-Whitney U-statistic through model sampling experiments from a normal population. The author infers that T_3 is likely to be more powerful than U for two-sided tests but U appears to be most powerful of all when a single tail alone is used. Some of the other statistics such as T_4 and W_3 (two-tailed) appear to be good competitors. Further, these statistics appear to be considerably more powerful when only one tail is used than when both the tails are taken into account.

The author demonstrates, with the help of examples, how the statistics W and T could be used in testing whether (i) a given sequence of observations is random; (ii) two samples come from the same population and (iii) two sets of observations arise from distributions differing in the location or a scale parameter.

Tables A-I and A-II give the probability distributions of W_3 and W_4 respectively for $n_1 = 3$, $n_2 = 1, 2, 3$; $n_1 = 4$, $n_2 = 1, 2, 3, 4$; $n_1 = 5$, $n_2 = 1, 2, 3, 4, 5, 6$ (n_1, n_2 being the sizes of two samples). Tables A-III and

SINGH, B. N. (Indian Standards Institutions, New Delhi) **5.3** (5.6) *continued*

On the application of the statistics W and T for testing two samples—*In English*

J. Indian Soc. Agric. Statist. (1958) **10**, 107-130 (8 references, 6 tables)

A-IV give the probability distributions of T_3 and T_4 respectively for $n_1 = 3, n_2 = 1, 2, 3$; $n_1 = 4, n_2 = 1, 2, 3, 4$; $n_1 = 5, n_2 = 1, 2, 3, 4, 5$. Table B-I gives the mean and standard deviations of W_3 and W_4 for $n = 20$, $n_1 = 10, 12, 15$; $n = 24$, $n_1 = 12, 16, 18$; $n = 30$, $n_2 = 15, 20, 24, 25$; $n_3 = 40$, $n_1 = 20, 30, 32, 35$; $n = 50, n_1 = 25, 30, 40, 45$; $n = 60, n_1 = 30, 40, 45, 50$ $(n = n_1+n_2)$. Table B-II gives similar figures for T_3 and T_4.

(Y. R. Rau)

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VESSEREAU, A. (University of Paris) **5.2** (–.–)

Sur les conditions d'application du criterium χ^2 de Pearson—*In French*

Bull. Int. Statist. Inst. (1958) **36**, III 87-101 (5 tables, 1 figure)

In order to see what happens at the " tails " of distributions, it may be useful to introduce the small theoretical values in Pearson's χ^2 test, even if it means losing some of the strictness of this test.

It happens, for example, that a distribution is well adjusted to a normal law in its central part, but not in its extremities. Grouping some classes may then make disappear, in an arbitrary way, anomalies which are very dangerous when the corresponding values interfere in security or health matters.

The three attempts made in trying to define the conditions of application of Pearson's χ^2 test lead to the following conclusions:

(1) The method of moments gives a very general method of recognising if the distribution of

$$X^2 = \sum_1^k (n_i - Np_i)^2/Np_i$$

is close to χ^2 with $(k-1)$ degrees of freedom. But this method is not very practical because it entails rather tedious numerical calculations; it is not known how the variations existing between moments of X^2 and moments of χ^2 should be interpreted, comparison of moments does not take directly into account the discontinuity of X^2.

The conclusion of this study is rather negative: it does not seem possible to define the conditions of a correct application of the χ^2 test with an easy and accurate rule applicable to every case.

(2) The study of a very particular case (when all the p_i are equal) shows that under certain circumstances, the test of χ^2 is of practical value, even if the numbers are small, even so small as to equal one in each class. In that particular case, it is the total amount of observations N which matters rather than the numbers in each class.

Besides, this study seems to prove that in the area of the distribution of X^2 which is covered by the usual tests (levels of 5 or 1 per cent.), the discontinuity does not introduce a very important error.

(3) After having pointed out that the distributions of X^2 and χ^2 tend to increasing difference as the theoretical law (characterised by probabilities p_i) is more " unbalanced ", the third study proposes a transformation

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VESSEREAU, A. (University of Paris)
Sur les conditions d'application du criterium χ^2 de Pearson—*In French*
Bull. Int. Statist. Inst. (1958) **36**, III 87-101 (5 tables, 1 figure)

5.2 (–.–)
continued

of the variable making the transformed variable and a χ^2 variable, of which the number of degrees of freedom is modified, coincide by their first two moments (expected value and variance).

This transformation is not to be recommended in practice, for it imposes tedious calculations. But even taking into account the statements summarised in (1) and (2), it suggests finally a rule, which is:

When all theoretical numbers are at least equal to a few unities, apply test of χ^2 under its usual form; when some of them are small—ranging about unity—consider that the limit value for X^2 at the level of 5 per cent. is the value found in the table of χ^2 with $(k-1)$ degrees of freedom at the level of 2 or 3 per cent.

(A. Vessereau)

VESSEREAU, A. (Report of discussion on a paper by)
On the application of Pearson's χ^2-test—*In English*
Bull. Int. Statist. Inst. (1959) **36**, I 111-113.

5.2 (–.–)

The contributors to this discussion were Gumbel, Hotelling, Cochran and Hemelrijk.

Gumbel drew attention to the point that the analysis of a continuous variate was dependent on the length of the class interval and the starting point of the classification. He referred to the work of Mann & Wald in this connection and urged the use of the probability integral transformation so that the end-points were always zero and unity.

Hotelling referred to the difficulties of using analytic number—theoretic methods in summing exact or appropriate probabilities over the lattice points within a large ellipsiod. An inequality for the difference between the Pearson approximation and the exact probability was on file in the North Carolina library in a thesis by Vora.

Cochran said that, while the operating rule proposed in the paper was both simple and reassuring, there were two situations where the exact moment of chi-squared might be required. These concern the analysis of a single multinomial and a *two by n* contingency table where many expectations were small although the overall number of degrees of freedom was large. In the case of the contingency table, Haldane's paper [*Biometrika* (1937) **29**, 389-391] gave the correct first four moments.

Hemelrijk referred to some work, using a method related to the Poisson index of dispersion, which avoided some of the difficulties in the case of only a small total number of observations and many empty classes. A report (in Dutch) by van Klinken & Prins was available from the Mathematical Centre, Amsterdam.

(W. R. Buckland)

WALSH, J. E. (System Development Corporation, Santa Monica, Cal.) **5.6** (5.2)

Non-parametric tests for median by interpolation from sign tests—*In English*

Ann. Inst. Statist. Math., Tokyo (1960) **11**, 183-188 (8 references)

The author treats non-randomised non-parametric tests for the median. He considers the case where the population distribution is continuous and symmetrical, and presents two interpolation methods to obtain a test with an approximate significance level to the desired one, α, without using the randomisation method. Denote the order statistic of the set of n independent observations by $x_1 \leqq x_2 \leqq \ldots \leqq x_n$, while by ϕ, the median of the population from which these observations were drawn. Now, let us consider a type of one-sided sign tests; accept $\phi > \phi_0$, if $x_i > \phi_0$. The significance level of this test is $\Pr(x_i > \phi) = \frac{1}{2} \sum_{j=0}^{i-1} \binom{n}{j}$. If α is the desired significance level and there does not exist any $i (0 \leqq i \leqq n)$ with $\Pr(x_i > \phi) = \alpha$, there exists i with $\Pr(x_i > \phi) < \alpha < \Pr(x_{i+1} > \phi)$. Then $[x_i, \Pr(x_i > \phi)]$ and $[x_{i+1}, \Pr(x_{i+1} > \phi)]$ are the end-points for the interpolation. For one interpolation method, the intermediate point $\left[\frac{x_i + x_{i+1}}{2}, P\left(\frac{x_i + x_{i+1}}{2} > \phi\right)\right]$ is added to the end-points. Here,

$$\Pr\left[\frac{x_i + x_{i+1}}{2} > \phi\right] = \Pr[\phi_{i+1} > \phi] - \binom{n-1}{i} / 2^n.$$

Using these three points and an appropriate interpolation method, the first co-ordinate value corresponding to the value α for the second co-ordinate is determined. Then, representing this value by $x_i(\alpha)$ we have $\Pr[x_i(\alpha) > \phi] \doteqdot \alpha$.

A different kind of interpolation consists in replacing x_i of the above test by a statistic of the form

$$\tfrac{1}{2}\min[2x_{i+1},\ x_i + ax_k + (1-a)x_{k+1}],$$

where $0 \leqq a \leqq 1$ and $i \leqq k \leqq n-1$. If the desired significance level α satisfies $\Pr\{\frac{1}{2}\min[2x_{i+1},\ x_i + x_k] > \phi\} \leqq \alpha \leqq \Pr\{\frac{1}{2}\min[2x_{i+1},\ x_i + x_{k+1}] > \phi\}$, the value of a to be taken in the test statistic is approximately evaluated by linear interpolation. For that the author employs the equality

$$\Pr\{\tfrac{1}{2}\min[2x_{i+1},\ x_i + x_t] > \phi\} = \Pr(x_{i+1} > \phi) - \binom{n+i-t}{i} \frac{1}{2^n}.$$

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(Y. Suzuki)

WEIR, J. B. de V. (University of Glasgow) **5.2** (3.1)

Significance of the difference between two means when the population variances may be unequal—*In English*

Nature (1960) **187**, 438 (7 references)

In this short communication the author presents a simple test (applicable at the five per cent. level) of the general case relating to the significance of the difference between two average values. This test is based upon a formula suggested by Welch [*Biometrika* (1938) **29**, 350-362] and is an extension of the standardised t-test already reported by the author [*Nature* (1960) **185**, 558: abstracted in this journal No. 392, **3.1**].

The standardised t, denoted t_s, may be written

$$t_s = (\bar{x}_1 - \bar{x}_2) \Big/ \left\{ \frac{\Sigma_1 + \Sigma_2}{n_1 + n_2 - 4} \left(\frac{1}{n_1} + \frac{1}{n_2} \right) \right\}^{\frac{1}{2}}$$

when $\bar{x}_1$ and $\bar{x}_2$ are the mean values of the two samples n_1, and n_2 and Σ_1 and Σ_2 the sums of squares about the mean values in these two samples.

The criterion (d) associated with Berens [*Landw. Jb.* (1929) **68**, 807-837] can also be adjusted as follows

$$d_s = (\bar{x}_1 - \bar{x}_2) / [\Sigma_1 / \{n_1(n_1 - 3)\} + \Sigma_2 / \{n_2(n_2 - 3)\}]^{\frac{1}{2}}.$$

The relationship between d_s and d was calculated and it was noted that at the five per cent. level the values of the d_s ranged between 1·96 and 2·00 (as was found for t_s) whereas those for d ranged between 1·960 and 2·447.

If $n_1 \neq n_2 > 6$ and $d_s \geqq 2$, then the two mean values are significantly different at the five per cent. level: if $d_s < 1{\cdot}96$ the difference is not significant. When $n_1 = n_2$ the adjusted Behrens criterion (d_s) is slightly more stringent. There is a factor $(n-3)$ in the denominator of d_s, compared with $(n-2)$ in t_s, which reflects the absence of information about the relative scale factors of the two populations.

(W. R. Buckland)

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BARTON, D. E. & DAVID, Florence N. (University College, London) **6.0** (4.2)
Models of functional relationship illustrated on astronomical data—*In English*
Bull. Int. Statist. Inst. (1960) **37**, III 9-33 (48 references, 1 figure)

This paper begins by outlining the astronomical problem of the gravitational field of our galaxy (in the neighbourhood of the solar system) as expressed in the relation between the angular velocity, w, and the radial distance, r, (referred to the galactic centre) of stars of certain type (supposed to be approximately linear but with a possible quadratic component). The nature of the errors in the determination of w and r is discussed.

The classical regression analysis, that is the analysis of n independent pairs of variables (y, x), is then outlined in the case where only the y-variable is random. A new technique called " generalised weighted least squares " is defined which is applicable to regressive situations where the y is heteroscedastic having second, and possible higher, moments as functions both of x and of the unknown regression parameters. The method consists of maximising, with respect to these parameters, the pseudo-likelihood function obtained by treating y as a normal variable with its proper mean and variance. This is shown, in outline only, to give consistent and asymptotically normal estimators. Heuristic argument suggests these are quite efficient. Analysis is next made of situations where both the observed variables consist of " underlying variables " to which have been added a " superimposed random error " under three headings:

(*a*) The Random Model; where the underlying variables are also random and the problem is thus reduced to the classical one of estimating parameters in a bivariate law. Or, if y is considered regressively and conditional on x, in univariate laws most commonly using 3 per cent. maximum likelihood estimators.

(*b*) The Parametric Model; where the underlying variables are considered to be parameters connected by a functional relation.

(*c*) Berkson's controlled variable model.

These models are discussed in relation to: experimental set-ups where they might be used; the choice of corresponding reference sets; the various regression functions (notably polynomial ones) and the various simplifying assumptions which have been made; for

BARTON, D. E. & DAVID, Florence N. (University College, London) **6.0** (4.2)
Models of functional relationship illustrated on astronomical data—*In English*
Bull. Int. Statist. Inst. (1960) **37**, III 9-33 (48 references, 1 figure)
continued

example a known fixed ratio between the standard errors of the superimposed errors. The authors say firstly that the " unidentifiability " impasse arises here only in the linear normal case and even so the least-squares estimate will only be negligibly biased in any real situation. The chief problem here is the contrast between computational intractability of the likelihood equations and the inefficiency of the method of moments (and other defects of various special methods.) The generalised weighted least-squares method provides estimators with reasonable efficiency and not too extensive computation. Secondly, the authors state that there are here an indefinitely large number of parameters as $n \to \infty$ and this has been shown to give " inconsistent " maximum likelihood estimators even of " structural " parameters. This is criticised as extending the definition of consistency (originally as the *number of replications* $\to \infty$) and it is argued that a parameter implies possibility of replication whereas an indefinite sequence of parameters is often made effective only by randomising.

Testing of regression hypotheses is briefly discussed and the data is used to illustrate the theory. Finally it is pointed out that a mixture of two bivariate populations can show regression that is apparently quadratic. An extensive bibliography is given on the topic of " structural relations ".

(D. E. Barton)

BARTON, D. E. & DAVID, Florence N. (Report of discussion on a paper by) **6.0** (4.2)
Models of functional relationships illustrated on astronomical data—*In English*
Bull. Int. Statist. Inst. (1960) **37**, I 69-71

Those taking part in the discussion were Geary, Mahalanobis and Zacks.

Geary made the general comment that the authors had dealt only with errors of measurement and that while aberrations or shocks probably existed in astronomy it was taken that they were small. In this connection it might be more realistic to assume the (unknown) error variance as monotonically increasing rather than independent of the x_i. The methods of moments or cumulants, referred to in section 3.1, was dismissed too summarily: there was also non-linear cumulant theory.

Mahalanobis proposed a method of analysis and asked whether it was of possible use. The method was based upon a large sub-sample of the n sub-periods into which the total observation period of time (T) was divided: the remaining sub-periods would form a second sub-sample. Hence it would be possible to calculate the regression parameters for the two sub-samples as well as for the whole set of data.

The method had the advantage that it was unnecessary to obtain a theoretical expression for the errors in the regression parameters and the results would be valid for most practical purposes the case in of sub-normal populations. In certain suitable circumstances it was possible to use more than two sub-samples by appropriate random partitioning of observations occurring in each sub-period.

(W. R. Buckland)

CORNFIELD, J., STEINFELD, J., & GREENHOUSE, S. W. (Nat. Inst. Health, Bethesda, Md.) **6.7** (0.5)
Models for the interpretation of experiments using tracer compounds—*In English*
Biometrics (1960) **16**, 212-234 (24 references, 4 figures)

This article attempts the formulation and analysis of a three-compartment model to explain the behaviour of albumin in the body of one experimental subject, using radioactive iodine (I^{131}) as tracer. One compartment is referred to as "Vascular", the second as "Extravascular", and the third as "Metabolised or Excreted". A reversible first-order reaction rate is postulated for the exchange between compartments 1 and 2, and irreversible first-order reaction equations from 1 and 2 to 3. The paper discusses various methods of obtaining the constants of the rate equations, and traces each method of determination back to a set of implicit or explicit assumptions.

The proposed statistical estimation is a direct least-squares approach, but sources of fallacies are discussed, especially multiple solutions which may arise in numerical iteration methods. The 'fit' of the rate equations to the data is of exceptionally good quality.

The authors suggest an extension of this method to more than three compartments. In this paper they present a detailed outline of solutions only for that situation in which flow to one "central" compartment is irreversible. Reference is also made to the especially simple method of difference equations (Prony) and to Hartley's method of "internal least-squares" [*Biometrika* (1948) **35**, 32-45]. Detailed numerical methods are not stated, but the final results have been reported. Estimations are based on measurements of concentration of labelled albumin in blood and in urine.

(R. E. Bargmann)

CHAKRAVARTI, N. K. (Defence Research Laboratory, Kanpur, India) **6.8** (4.4)

The mathematical theory of biological assay of a local anaesthetic—*In English*

Biometrics (1960) **16**, 278-291 (10 references, 3 tables)

No satisfactory method exists for biologically assaying the potency of a local anaesthetic. Many forms of relationships are used and the essentially arbitrary nature of all of them have led Gray & Geddes (*J. Pharm. Pharmacol.* (1954) **6**, 89-114] to state that there is confusion of methods.

The purpose of this paper is to formulate a theoretically sound approach which can alleviate this confusion so that the vast accumulation of data from research laboratories throughout the world can be compared and put to use.

The author summarises the review paper of Gray & Geddes which deals with the pharmacologic action of local anaesthetics. The important point here is that local anaesthetics are more soluble in lipoids than in water. Nerve tissue is rich in lipoids, consequently local anaesthetics easily enter into the lipoid rich plasma-membrane, which is a specialised membrane of lipoid and protein, perhaps only two molecules thick. It is due to the thinness of the membrane that the concentration of anaesthetic is of consequence rather than the amount injected for the thin membrane is capable of dissolving only a small amount of anaesthetic.

In the theory section the author discusses three types of observations and their respective relationships in a bioassay in which an unknown concentration is assayed against a set of known concentrations. The three types are:

(i) a set of n stimuli applied at each of m selected intervals of time $t_1, \ldots, t_m$ per concentration per animal.

(ii) a set of n stimuli applied after one fixed interval of time t per concentration per animal.

(iii) Duration of anaesthesis obtained by applying stimuli in succession so that the middle point of time between the last failure to perceive and the first perception is taken as the duration of anaesthesia.

The author discusses the fitting of the relationships for the three cases above. He also discusses placing confidence limits on C, the unknown concentration, for each case.

CHAKRAVARTI, N. K. (Defence Research Laboratory, Kanpur, India) **6.8** (4.4) *continued*

The mathematical theory of biological assay of a local anaesthetic—*In English*

Biometrics (1960) **16**, 278-291 (10 references, 3 tables)

Using data from Bülbring & Wajda [*J. Pharmacol.* (1945) **85**, 78-84] the author presents experimental verification of the theory for case (i).

For case (iii), where duration of anaesthesia is considered as a criterion, the author presents ten empirically fitted relationships by other writers. With some calculations he points out that his relationship derived from theoretical considerations shows excellent agreement with the observed data reported by these authors. The numerous forms of relationships assumed arbitrarily are no longer necessary and a uniform theoretically valid relationship replaces all the arbitrary ones successfully.

The theory also shows that there is no fundamental difference between criteria based on duration of anaesthesia and degree of anaesthesia. However, the criterion based on degree of anaesthesia is expected to lead to more precise results since degree is generally based on many observations, whereas duration is a mean of two observations.

(J. J. Bartko)

KUDO, A. & FURUKAWA, N. (Kyushu University, Japan) **6.8** (4.3)

A model in probit analysis—*In English*

Bull. Math. Statist. (1958) **8**, 1-7 (5 references)

The probit analysis treated in this paper is for the case that the reaction caused by the stimulus is composed of only two, " response " or " non-response "; for example, death or survival. The minimum intensity of the stimulus to result in " response " is called the tolerance of the subject under application of the stimulus. The problem considered in this paper is the investigation of the case when the number of subjects under application is not known. Two cases are considered: (*a*) the case where the number of individuals is very large and (*b*) the case where that number is not so large. In the case (*a*), the distribution of the tolerance of the group under consideration can be assumed as Fisher's limit distribution of the maximum of independent random variables. The authors assume for the case (*b*) that the number, n of individuals in each experiment is distributed according to the Poisson law. The distribution of the tolerance of the group is taken as

$$\sum_{n=0}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} [F(x \mid \theta)]^n = e^{-\lambda[1-F(x \mid \theta)]},$$

where λ is the average number of individuals and $F(x \mid \theta)$ is the tolerance distribution of individuals.

The method for calculation of the maximum likelihood estimates of the parameters is explained.

(M. Siotani)

LINHART, H. (S. African C.S.I.R., Johannesburg) **6.1** (4.4)

A measure of predictive precision in regression analysis—*In English*

Ann. Math. Statist. (1960) **31**, 399-404 (10 references)

The problem which the author considered here is as follows: Suppose $\boldsymbol{l}_\nu (\nu = 1, 2, \ldots, n)$ are n independent $(k+1)$-dimensional vectors which are identically distributed with mean μ and having an unknown positive-definite covariance matrix $\boldsymbol{\Lambda}$. An $(n+1)$st vector is considered, and it is presumed that only the last k components of this vector are known. Using the information obtained from the first n vectors about the underlying vector population from which these vectors are samples, one is required to construct a confidence interval estimate for the missing component of the last vector. Since the quantity being estimated is not a parameter, but a future observation, the interval is more properly a " prediction interval "; this is noted by the author.

The confidence interval is derived and is shown to be unbiased in the sense of Neyman. A discussion of the shortness of the interval, in the sense of Neyman, is discussed. The distribution of the physical length of the confidence intervals is derived, and from this a formula for the expected length is obtained.

This paper is a follow-up and discussion of an earlier paper by the same author [*Psychometrika* (1960) **25**, 45-58; abstracted in this journal No. 627, **5.8**].

(D. H. Shaffer)

LORD, F. M. (Educational Testing Service, Princeton, N.J.) **6.1** (6.2)
Problems in mental test theory arising from errors of measurement—*In English*
J. Amer. Statist. Ass. (1959) **54**, 472-479 (13 references)

In this paper the author presents a number of basic open problems in mental test theory, he explains their logical status and offers some tentative suggestions as to how solutions for some of these might be obtained. The object of the paper is not only to inform but also to stimulate critical thought and the development of theory.

The author first defines the notions of actual score, true score, and error of measurement. He then goes on to discuss the commonly made assumption that the errors of measurement are normally and independently distributed with an equal variance about the regression curve of the true scores. The assumption of homoscedasticity, in particular, is pointed out to be often unrealistic as, for example, when the appropriate model is the item-sampling binomial one.

With regard to estimating an examinee's true score, the commonly used estimator is not the actually observed score but that score regressed linearly on the observed mean of the group to which the examinee belongs. It is pointed out that the best regression curve is not necessarily linear and that procedures for the non-linear situation are not at present at the disposal of the psychologist.

The author gives a brief discussion of the problem of estimating the frequency distribution of true scores and suggests that a possible approach is through the definitional integral equation linking the distribution of true scores with the marginal and conditional distributions of actual scores.

Consideration is given to the problem of determining whether two tests measure the same characteristic. A satisfactory method for estimating the size of the correlation between true scores on the two tests exists only when that correlation is linear. The author suggests that a test of the hypothesis that the true scores are functionally related can be developed by exploiting the bivariate analogue of the equation mentioned in the preceding paragraph.

Finally, the logical structure of a possible test for the measurement of change is suggested as that contrasting the observed bivariate distribution of the scores on the two consecutive occasions with that which would be expected had no change occurred.

(G. Zyskind)

MARTIN, L. (Free University, Brussels) **6.8** (0.2)
Analysis and fitting of unimodal curves by the probit method—*In French*
Bull. Int. Statist. Inst. (1958) **36**, III 43-58 (45 references, 3 tables, 5 figures)

In the introduction to this review paper the author makes a brief review of four methods for fitting unimodal curves and then follows a percentile method similar to that used by other writers including Edgeworth, Kapteyn & van Uven, Johnson, Cornish and Fisher. The present author, however, introduces a generalised normal deviate in terms of orthogonal polynomials whose coefficients are calculated from a (probit) weighted regression.

The second section reviews certain classical methods for transforming curves and the following sections (three and four) deal with the linear and non-linear probit transformations. This discussion prepares the way for the author's contribution using orthogonal polynomials. An " indicator curve", for which an expression is given, translates a fixed standard normal distribution curve into the unimodal frequency curve under analysis. Several types of this indicator curve are given together with an application of the method to well known data from Pretorius' work [*Biometrika* (1930) **22**, 109-223]. It is shown that the fit is better than that achieved by a Pearson Type IV curve and as good as that given by the method originally proposed by Johnson [*Biometrika* (1949) **36**, 149-176] or as modified by Draper (*Biometrika* (1952) **39**, 290-301].

(W. R. Buckland)

PHILLIPS, A. W. & QUENOUILLE, M. H. (London School of Economics) **6.6** (–.–)

Estimation, regulation, and prediction in interdependent dynamic systems—*In English*

Bull. Int. Statist. Inst. (1960) **37**, II 335-343 (6 references)

The statistical analysis of interdependent dynamic systems has developed separately in different fields of study, particularly in control engineering and in economics. Control engineers have usually formulated their theories in terms of differential equations, giving careful attention to continuously distributed lagged dependences and other forms of dynamic relationship. Economists have tended to formulate their theories in terms of difference equations, partly because some discrete adjustments do occur in economic systems but often for the less satisfactory reason that their observations are taken at discrete points of time or over discrete time intervals. Both continuous and discrete systems are considered in this paper.

In econometric work predictions are often made of the future values of certain variables in the economic systems, on the assumption that the system and the stochastic properties of the disturbances to it remain unchanged. In both control engineering and economics it is usually more useful to consider conditional predictions, on various assumptions concerning the values of those variables which are subject to direct control, or predictions about the changes in the operating performance of the system which would result from a modification of its structure. Such predictions require knowledge of the structural equations and so raise the problems of estimation and identification. In both continuous and discrete systems the problem of identification will often be overcome if careful attention is given to the formulation of the basic model as a system of dynamic behaviour relationships.

The problem of optimising the performance of a system by suitable modification of its structure is illustrated with reference to a simple model of economic regulation.

(A. W. Phillips)

POMPILJ, G. (University of Rome) **6.0** (6.1)

On entirely pseudo-linear regression—*In English*

Bull. Int. Statist. Inst. (1958) **36**, III 60-63 (10 references)

If the regression function between two quantitive phenomena is linear then all least-squares polynomials with which to approximate the regression function are also linear. It has been shown by Dall'Aglio [*R.C. Mat. Univ. Roma* (1956)] that the proposition is not invertible and that when all least-squares polynomials are linear the regression will be " pseudo-linear ". According to the author two random variates have an " entirely " linear regression whatever the direction of the axis. This paper combines these two concepts: in particular it considers two random variates with the origin at $(\bar{x}, \bar{y})$, $\sigma_2 = \sigma_y = 1$ and $\sigma_{xy} = 0$.

Five theorems are given for the characterisation of entirely pseudo-linear regression—the third theorem is also given in a form which does not rely upon certain transformations relating to the affine theory of random variables.

(W. R. Buckland)

QUANDT, R. E. (Princeton University)

6.1 (5.1)

Tests of the hypothesis that a linear regression system obeys two separate regimes—*In English*

J. Amer. Statist. Ass. (1960) **55**, 324-330 (14 references, 3 tables)

This paper is concerned with testing the hypothesis that a change has occurred in the parameters of a linear regression system. Several possible tests are presented, each hypothesising that no shift has occurred, with the single alternative being that one shift has occurred.

In a previous paper [*J. Amer. Statist. Ass.* (1958) **53**, 873-880; abstracted in this journal No. 249, **6.1**] the author had derived the likelihood ratio test for the above-mentioned hypothesis, and conjectured that $-2 \log \lambda$ for large samples might be approximately distributed as a chi-square variable with four degrees of freedom. This conjecture was investigated empirically for three different sample sizes and found to be false. However, the empirical distribution of $-\log \lambda$ is given in tabular form and the estimated probability of $-\log \lambda$ being less than various values K was presented. These may be used to perform an empirical test of the hypothesis of interest.

Three small sample tests are presented and discussed. The first of these is approximately a Student's t-test. The test criterion, however, is dependent upon an estimate of when the shift in parameters occurred, thus making its exact distribution unknown. A second difficulty with this test is that the expected value of the test statistic can equal zero under the alternative hypothesis as well if a certain relation among the parameters holds.

A second small sample test also makes use of the Student's t-distribution. In this test the switching point is arbitrarily estimated to be at the middle observation. It is feared, however, that the power of the test is quite dependent on how close the true switching point is to the middle observation. The third small sample test again arbitrarily chooses the middle observation as the switching point and utilises an F-variable. It also is subject to criticism in that again the power depends on how close the true switching point is to the centre observation.

(H. Larson)

RIOS, S. (Institute of Statistical Research, Madrid)
Sur les lignes de regression—*In French*
Bull. Int. Statist. Inst. (1958) **36**, III 64-70 (1 reference)

6.1 (–.–)

Regression curves are usually defined as curves in the neighbourhood of which the scatter diagram is condensed. In this paper the author starts from this idea in defining a regression line.

Consider a two-dimensional random variable (ξ, η), whose frequency function is $f(x, y)$, and also a curve L of the equation $y = g(x)$ and the strip B of width 2ε, limited by the curves $y = g(x) \pm \varepsilon$.

Put:

$$H[g(x), \varepsilon] = \iint_B f(x, y)dxdy$$

and let it be supposed that the following limit exists

$$\lim_{\varepsilon \to 0} H[g(x), \varepsilon]/2\varepsilon = G[g(x)].$$

The function $g(x)$, which makes this functional a maximum gives us the regression line of η on ξ.

It is shown, then, that the regression curve of η on ξ makes the integral $\int_L f(x, y)dx$ a maximum. If $M_2(x)$ is the mode of the distribution of η conditioned by $\xi = x$, the before-mentioned regression line coincides with the curve $y = M_2(x)$.

In the same way, modal regression is a straight line of η on ξ, to which corresponds a maximum density for a strip in the neighbourhood of the line arbitrarily small in width in the direction of the axis y.

It is shown that in this method the parameters α, β of the regression straight $y = \alpha + \beta x$ should satisfy the equations:

$$\frac{\partial G}{\partial \alpha} \equiv \int_{-\infty}^{\infty} \frac{\partial f(x, \alpha+\beta x)}{\partial(\alpha+\beta x)} dx = 0;$$

$$\frac{\partial G}{\partial \beta} \equiv \int_{-\infty}^{\infty} \frac{\partial f(x, \alpha+\beta x)}{\partial(\alpha+\beta x)} xdx = 0.$$

In the case of bivariate normal distribution these regression straights are the same as the mean square regression straight lines.

The result gives two graphical methods for obtaining the classical regression lines for a normal scatter diagram.

RIOS, S. (Institute of Statistical Research, Madrid)
Sur les lignes de regression—*In French*
Bull. Int. Statist. Inst. (1958) **36**, III 64-70 (1 reference)

6.1 (–.–)
continued

In the same way the parabolic regression curves are defined and the condition equations:

$$\frac{\partial G}{\partial \alpha_0} \equiv \int_{-\infty}^{\infty} \frac{\partial f}{\partial y} dx = 0;$$

$$\frac{\partial G}{\partial \alpha_1} \equiv \int_{-\infty}^{\infty} \frac{\partial f}{\partial y} xdx = 0; \ldots;$$

$$\frac{\partial G}{\partial \alpha_n} \equiv \int_{-\infty}^{\infty} \frac{\partial f}{\partial y} x^n dx = 0$$

for the parameters $\alpha_0, \alpha_1 \ldots \alpha_n$ of the parabola $y = \alpha_0 + \alpha_1 x + \ldots + \alpha_n x^n$ are obtained.

Finally, these results are generalised. A functional $D[\xi, c]$ is defined and named *dispersion* of the random variable ξ in the point c. With fixed values of ξ, the functional reaches a minimum for $c = c_1$. In this case call the *central value* of ξ number c_1 and write $c_1 = C[\xi]$.

In this way a generalised regression line is defined and a minimising property is shown.

(S. Rios)

SCHNEEBERGER, H. (Wiesbaden) **6.1** (5.2)

Theory and applications of tests on the hypothesis of linearity in total regression equations. I—*In German*

Math., Tech., Wirtschaft (1960) **3**, 118-122 (6 references, 2 figures)

A theorem is formulated and partially proved, by which the hypothesis of linearity $H_0 : \mathscr{E}y = \alpha+\beta(x-\bar{x})$ may be tested assuming (*a*) x non-random, (*b*) y normal, (*c*) $\mathscr{E}(y-\mathscr{E}y)^2 =$ constant for each x. Another test which became standard practice in statistical text-books some years ago [see, for example, Linder, *Statistische Methoden*. (1945) *Birkhäuser, Basel*] is based on the same assumptions and the additional one that there exists at least one y value to which two or more nonidentical y values correspond. The new test, however, is free from this additional assumption.

Generalisation is obtained by grouping the data for analysis of variance technique in a way that a certain group of y values will correspond either to one single x value (T_1 type groups) or to some interval on x (T_2 type groups). In practice, T_2 type groups will contain those y values which are exclusive partners of certain x values. Then H_1 means that the expectation within a T_1 type group of y values is a point, the expectation within a T_2 type group of y values is a straight line in the x, y-plane, and not all of these points and straight lines lie on the other straight line defined by H_0.

Let n be the number of points x, y. If the following quantities are given;

(i) m_k the number of T_k type groups of y values selected ($k = 1, 2$),

(ii) $m_1+m_2 = m$ the number of all groups of y values together,

(iii) y_{ij} the ith y value ($i = 1, 2, \ldots, n_j$; $\sum_j n_j = n$) in the jth group ($j = 1, 2, \ldots, m$) of y values,

(iv) y_{ij}^* the regression estimate of y_{ij} based on all n points $\{x, y\}$ by means of linear regression equation,

(v) y_{ij}^{**} either the arithmetic mean of all y values in the jth group if of T_1 type, or the regression estimate of y_{ij} based on the n_j points $\{x, y\}$ in the jth group by means of linear regression equation if of T_2 type,

then, assuming (*a*), (*b*) and (*c*), $F = s_1^2/s_2^2$ with

$$s_1^2 = \sum_{i,\,j} (y_{ij}^*-y_{ij}^{**})^2/(m_1+2m_2-2)$$

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SCHNEEBERGER, H. (Wiesbaden) **6.1** (5.2) *continued*

Theory and applications of tests on the hypothesis of linearity in total regression equations. I—*In German*

Math., Tech., Wirtschaft (1960) **3**, 118-122 (6 references, 2 figures)

and $$s_2^2 = \sum_{i,\,j} (y_{ij}-y_{ij}^{**})^2/(n-m_1-2m_2)$$

is distributed according to the F-distribution with $f_1 = m_1+2m_2-2$ and $f_2 = n-m_1-2m_2$ degrees of freedom (part I of the theorem). H_0 will be rejected at the P per cent. level of significance if the observed F is equal to or larger than the tabulated value with parameters f_1, f_2 and P (part II of the theorem).

The author announces a second part of his study with a further generalisation of the theorem already given. This generalisation will consist in a qualification of assumption (*c*) by permitting different values $\mathscr{E}(y-Ey)^2$ for the various groups, assuming constance within (T_2 type) groups only. Finally, as an application of the theorem in part II, a third part of the study will include a test which will be worked out for the hypothesis that more than two regression lines are stochastically identical. This test will be a multiple and simultaneous version of the standard t-test for the comparison of two total regression coefficients.

(R. K. Bauer)

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Values for filial and fraternal correlations for matings in a panmictic population are currently known only for the case of two alleles. The present paper shows how a correlation derived from actual experimental data may be compared with a theoretical correlation based on some genetic hypothesis when more than two alleles are involved. While it is claimed that most of the results hold for n alleles, the illustrative examples are given for $n = 3$.

The well known procedure of assigning weights 0, 1, 2 to the genotypes A_1A_1, A_1A_2, A_2A_2 for computing correlations between relatives when only two alleles are present does not satisfactorily extend to cases involving a larger number of alleles. When $n>2$, and using this method of mathematical weighting, it is impossible to assign a weight to each heterozygote which is intermediate to the weights assigned to the corresponding homozygotes.

The author solves this problem by employing a system of hyper-complex numbers. He locates the n homozygotes at the vertices of a regular simplex in an $(n-1)$-dimensional space and places the heterozygotes at the mid-points of the appropriate edges of the simplex. The origin of co-ordinates is taken at the centroid of the simplex. Thus for $n = 3$, the homozygotes are at the vertices of a triangle and the heterozygotes at the mid-points of the sides of the triangle. The weights used are the vectors joining the origin and the respective zygotic combinations. Thus A_iA_i has the weight V_i (magnitude 1) and A_iA_j has the weight $\frac{1}{2}(V_i+V_j)$. The scalar product of vector algebra is used in the ensuing manipulations, i.e., $V_i^2 = 1$, $V_iV_j = -1/(n-1)$, etc.

Two alternative approaches to the weighting problem are considered. These are based on the same principle but take the origin of co-ordinates at different positions. The author states his aim to be the development of the theoretical correlation coefficients and does not consider the problem of determining whether the sampling distribution of the sample correlation coefficient is independent of gene frequencies.

The method is applied in a series of illustrative examples. These include parent-child and sib-sib correlations, correlations in the case of sex-linked genes,

and correlations in the case of dominance. This last case is subdivided into:

Type I—A dominates B and C, B dominates C;

II—A dominates B and C, neither B nor C dominates;

III—A and B dominate C, neither A nor B dominates the other.

All illustrations are for the case of three alleles.

(M. F. Peck)

ABT, K. (U.S. Naval Weapons Lab., Dahlgreen, Virginia) **7.6 (6.1)**
Analysis of covariance and analysis of differences—*In French*
Metrika (1960) **3**, 26-45 and 95-116 (23 references, 5 tables, 1 figure)

At least two methods are available for the purpose of analysing paired data $(x;\ y)$ which are classified in one or more directions: where x and y are observations of the same variable and therefore usually observed at different times. The object of this paper is to compare the two methods. Firstly, there is the analysis of covariance in which y is considered to be stochastically dependent on x the independent variable. A second method is the analysis of variance of the generalised differences $d = y - \beta_0 x$, briefly called " analysis of differences ". The case $\beta_0 = 1$ represents the most important case of ordinary differences.

The first part of this study [published in two consecutive issues of *Metrika* (1960) **3**] deals with the basic principles and some special problems of the analysis of covariance. Reference is especially made to the paper by Fairfield Smith [*Biometrics* (1957) **13**, 282-308]. The discussion in this present paper is restricted to the case of one independent variable x.

It is usually stated that, in order to be able to apply the analysis of covariance, the differences between the " treatment " means of the independent variable x must not be significant. However, it is shown that even in this case the method is useful. A two by two table is given showing the conclusions which might be drawn from the data depending on the significance of non-significance of the differences among both the adjusted and the unadjusted treatment means of the dependent variable y.

Special attention is given to the decomposition of the so-called reduced sum of squares for the numerator of the F-test into the part which measures the deviations of the treatment means from their own regression and the second part which represents the contrast between the regression coefficient for treatment means and that for error. It is shown that this decomposition is useful in explaining the meaning of the null hypothesis being tested by the " reduced sum of squares for treatments ". Then, for the case of significant differences between the treatment means of the independent variable the conditions are given under which the two sums of squares might be used separately for testing certain hypotheses

ABT, K. (U.S. Naval Weapons Lab., Dahlgreen, Virginia) **7.6 (6.1)**
Analysis of covariance and analysis of differences—*In French*
Metrika (1960) **3**, 26-45 and 95-116 (23 references, 5 tables, 1 figure)
continued

concerning them. Finally, a table of symbols used in the analysis of covariance by various authors is given.

(K. Abt)

Remark of the Regional Editor:
The second part of this study will be published in *Metrika* (1960) **3**, part 3. It will discuss the relations between the analysis of covariance and the analysis of differences.

(J. Pfanzagl)

ANDERSON, R. L. (Institute of Statistics, North Carolina State College, Raleigh) **7.2** (7.3)

Uses of variance component analysis in the interpretation of biological experiments—*In English*

Bull. Int. Statist. Inst. (1960) **37**, III 71-90 (22 references, 7 tables)

Variance component analysis has as its main objective the assessment of sources of variability in experimental data. Hence it is a valuable tool in the use of an existing set of data to help in planning and conducting future experiments. Examples are given in this paper of studies on the sources of variability in determining the ascorbic acid content of turnip green leaves, in the production of an antibiotic such as streptomycin, and in determining the phosphorus content of chicken embryos. It is shown that a non-balanced sampling scheme may be desirable in order to obtain efficient estimates of all variance components.

Particular attention has been paid to the use of variance component analysis in quantitative genetics studies. In this case certain simple functions of the components are important genetic parameters. A major problem of the experimenter is to separate genetic and environmental sources of variability, because expected progress due to a breeding programme depends on a knowledge of the relative importance of the genetic sources.

Variance component analysis is often useful in determining error-mean-squares to test specific hypotheses. Difficult statistical problems arise in testing hypotheses regarding genetic and other parameters and in devising confidence limits for them; many of these problems remain unsolved at this time. Methods of analysis for the mixed model are considered, with examples of split-plot experiments.

The author states that many other theoretical problems in connection with variance component analysis should be investigated, such as the methods of estimation for non-balanced designs; optimal designs efficiently to estimate many components from the same data; effects of non-normality on the estimation and testing procedures and methods to alleviate the difficulties.

(R. L. Anderson)

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DWASS, M. (Northwestern University, Chicago, Ill.) **7.8** (7.7)

Multiple confidence procedures—*In English*

Ann. Inst. Statist. Math. (1959) **10**, 277-282 (5 references, 1 table)

In this paper the author deals with confidence interval statements for contrasts. By a contrast he means the quantity $\Sigma a_i\Delta_i$ with $\Sigma a_i = 0$, where Δ_i are some constants such that for independent random variables X_1, X_2, ..., X_n, which we observe, the random variables $X_1-\Delta_1$, ..., $X_n-\Delta_n$ have the same distribution. Assuming that $X-\Delta$ has a normal distribution $N(0, \sigma^2)$, the author obtains the following results: for any $p(1\leqq p\leqq\infty)$, we have confidence intervals

$$|\Sigma a_i\Delta_i-\Sigma a_iX_i| \leqq \| a-\hat{a}_q \|_q T_p$$

and

$$|\Sigma a_i\Delta_i-\Sigma a_iX_i| \leqq \| a \|_q T'_p$$

where $(1/p)+(1/q)=1$ and $\| a-\hat{a}_q \|_q = \inf_c \sum_{i=1}^{n} |a_i-c|^q$ or $= \inf_c \max_i |a_i-c|$, $\| a \|_q = (\Sigma_i |a_i|^q)^{1/q}$ for $1\leqq q<\infty$ or $= \max_i |a_i|$ for $q=\infty$, and T_p, T'_p are the upper $1-\alpha(0<\alpha<1)$ points of the distributions of $\|X-X\|_p/\sqrt{k}S$, and $\| X-X_p \|_p/\sqrt{k}S$ respectively, where $k(S/\sigma)^2$ is a k-degree of freedom chi-square variable which is independent of X. The probability that the above relation holds true for all contrasts is $1-\alpha$. The author also refers to some comparison of parameters Δ_i.

(H. Hudimoto)

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BECHHOFER, R. E. (Cornell University, Ithaca, N.Y.)

7.3 (9.0)

A multiplicative model for analysing variances which are affected by several factors—*In English*

J. Amer. Statist. Ass. (1960) **55**, 245-264 (24 references)

In this paper the author proposes a way of looking at multi-factor experiments which are conducted to study the effect of changes in the levels of the factors on the variance of a chance variable. A mathematical model is proposed which assumes that the effect of changing a factor from one level to another is to multiply the original variance of the chance variable by a positive constant. This multiplicative model allows one to speak of relative (percentage) change in the variance of the chance variable.

Suppose that one is considering rcN independent observations X_{ijk} in a two-way layout. The uni-variate multiplicative model for variances which is presented is:

$$\sigma_{ij}^2 = \sigma^2 \alpha_i \beta_j \gamma_{ij} \quad (i = 1, 2, \ldots, r; j = 1, 2, \ldots, c)$$

where

$$\sigma^2 = \left\{ \prod_{i=1}^{r} \prod_{j=1}^{c} \sigma_{ij}^2 \right\}^{1/rc}; \quad \alpha_i \sigma^2 = \left\{ \prod_{j=1}^{c} \sigma_{ij}^2 \right\}^{1/c};$$

$$\beta_j \sigma^2 = \left\{ \prod_{i=1}^{r} \sigma_{ij}^2 \right\}^{1/r}; \qquad \gamma_{ij} = \sigma_{ij}^2 / \alpha_i \beta_j \sigma^2.$$

The multiplicative model for variances shown above can be viewed as a direct analogue of the additive model for means and can be easily generalised so that experiments involving more than two factors can be analysed in conjunction with experimental patterns such as randomised complete blocks, Latin squares, balanced incomplete blocks, factorial arrangements etc. This model is appropriate when the factors under study are quantitative, qualitative, or both. However, when all of the factors which are being studied are quantitative, a more meaningful analysis will probably be obtained with the following approximating multiplicative model:

$$\mathrm{Var}(X_{a,b,k}) = \sigma_{a,b}^2 = \prod_{i=0}^{m_a} \prod_{j=0}^{m_b} (\lambda_{ij})^{a^i b^j}$$

$$(a \in A, b \in B, k = 1, 2, \ldots, N)$$

where $\lambda_{i,j} > 0 \; (i = 0, 1, \ldots, m_a; j = 0, 1, \ldots, m_b)$.

The author discusses the procedures for obtaining point estimates of the parameters, estimating the parameters individually and jointly by means of confidence

BECHHOFER, R. E. (Cornell University, Ithaca, N.Y.)

7.3 (9.0)
continued

A multiplicative model for analysing variances which are affected by several factors—*In English*

J. Amer. Statist. Ass. (1960) **55**, 245-264 (24 references)

regions, and making exact multiple comparison statements about the σ_{ij}^2. Some of the simpler forms of a univariate multiplicative model for variances are considered for the first multiplicative model and typical hypotheses that might be tested are indicated. It is then shown how these various hypotheses can be regarded as special cases of a uni-variate general multiplicative hypothesis. Finally, a multivariate generalisation of the uni-variate multiplicative model for variances for the two-factor complete factorial experiment with no interaction is considered and typical hypotheses that might be tested are indicated.

(S. Addelman)

KAPUR, M. N. (Indian Council of Agricultural Research, New Delhi) **7.7** (5.4)

On a class of decision procedures for ranking normal populations according to their variances—*In English*

J. Indian Soc. Agric. Statist. (1958) **10**, 99-107 (2 references)

In the present paper, a similar problem for variances is considered to that dealt with by Seal [*Ann. Math. Statist.* (1955) **26**] who obtained a class of decision procedures for choosing the smallest group of populations from a given finite set of normal populations so that the selected group includes the population with the largest mean. The author gives a decision procedure for dividing a given set of normal populations into two groups 1 and 2 such that group 1 contains the population with the largest variance.

The procedure given by the author has the property of unbiasedness, that is, the probability of excluding any population not with the largest variance from group 1 is not less than the probability of excluding the population with the largest variance. It also has the property of gradation, that is, the existence of a constant σ^2_{0r} corresponding to any $r(0<r<1)$ such that the chance of retaining a population with the variance σ^2_0 in group 1 is greater or less than r according as σ^2_0 is greater or less than σ^2_{0r}.

Finally, the author considers a special case in which the given set of $k+1$ populations are such that all populations except one have equal variance. A decision procedure for this case is obtained and proved to be optimum in the sense that

(i) it maximises the probability of retaining the population with the unequal variance in group 1 if its variance happens to be larger than the common variance of the remaining k populations;

(ii) it maximises the probability of not retaining the population with the unequal variance in group 1 if this is smaller than the common variance of the remaining populations.

(Y. R. Rau)

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KONIJN, H. S. (University of Sydney) **7.7** (7.1)

Multiple comparison with controls—*In English*

Aust. J. Statist. (1960) **2**, 16-18 (3 references).

The author's paper [*Aust. J. Statist.* (1959) **1**, 57-68] discussed methods of obtaining confidence limits for all possible comparisons. It often happens that the only comparisons of interest are comparisons with a control. This important case, which allows narrower confidence limits to be constructed, has been treated by Dunnett [*J. Amer. Statist. Ass.* (1955) **50**, 1096-1121] for the simple case of a one-way classification. In the present paper, Dunnett's method is applied to more complicated analyses of variance. The procedure is illustrated by applying it to an example given in the author's earlier paper, assuming a " fixed " model and not necessarily vanishing interactions.

(H. S. Konijn)

BARNARD, G. A. (Imperial College, London) **8.9** (10.2)

Control charts and stochastic processes—*In English*

J. R. Statist. Soc. B (1959) **21**, 239-257 (3 references, 1 table, 2 figures)

This paper contains a discussion, partly theoretical and partly empirical, on the use of the control chart procedures in estimating the value of a process mean. Firstly, there is a discussion of the use of control charts of cumulative sums. An empirical method is described based on a method proposed by Page [*Biometrika* (1954) **41**, 100-114 and *Biometrika* (1957) **44**, 248-252], of applying control limits to such charts. If μ is the required mean, and $x_1, x_2, \ldots, x_n, \ldots$, a sequence of observed sample means from samples taken at equal intervals of time, a, the procedure described is equivalent to applying control limits

$$S_n - d + ja) \tan\theta < S_{n-j} < S_n + (d + ja) \tan\theta,$$

where $s_r = \sum_{i=1}^{r} (x_i - \mu)$; d and θ are to be chosen by the statistician. The author suggests that control limits given by a parabolic boundary might sometimes be used; however, he does not consider the advantage likely to be marked. He is also of the opinion that exact probabilities of crossing control limits need not be computed.

In the second part of the paper a process is considered in which there are sudden jumps in the true value of the process mean. The sizes of these jumps are assumed to be independently normally distributed with zero mean and a known variance, σ^2. The jumps are assumed to occur randomly and independently in time, so that $\Pr\{r \text{ jumps in unit time}\} = e^{-\lambda}\lambda^r/r!$, λ is supposed known. The observed values are assumed to be independently normally distributed about the current process mean with unit variance.

The problem of estimating $\mu(t)$, the process mean at time t, is discussed. The best linear unbiased estimator is obtained first. It is suggested that a good approximation to this estimator would be provided by the exponentially weighted estimator

$$y_n = (x_n + Ax_{n-1} + A^2x_{n-2} + \ldots)/(1 - A)$$

where $A = 1 - \sigma\sqrt{\lambda}$. Since $y_n = (1-A)^{-1}x_n + Ay_{n-1}$, this estimator is well suited for continuous routine calculations.

The author proposes as an alternative, and probably better estimator, the mean likelihood estimator obtained by averaging with weight proportional to the likelihood

BARNARD, G. A. (Imperial College, London) **8.9** (10.2) *continued*

Control charts and stochastic processes—*In English*

J. R. Statist. Soc. B (1959) **21**, 239-257 (3 references, 1 table, 2 figures)

function of the observed values $x_1, x_2, \ldots, x_n$. He gives a heuristic justification for the assumption of a uniform prior distribution of μ; if this assumption is correct, the mean likelihood estimator does have a minimum variance. This estimator is a weighted mean of the x_i's, with weights depending upon the differences $x_{i+1} - x_i$ between successive observations. A practical graphic method for estimating μ is also described. Finally, the problem of estimating λ and σ, if they are not known, is discussed.

(N. L. Johnson)

BARNARD, G. A. (Report of discussion on a paper by)
Control charts and stochastic processes—*In English*
J. R. Statist. Soc. B (1959) **21**, 257-271 (1 figure)

8.9 (1.02)

Those taking part in the discussion were Cox, Jessop, Jowett, Stevens, Winsten, Jenkins, Durbin, Ewan, Moore, Eisenhart and Page (in writing): the author replied.

Cox drew attention to the characteristic of the model chosen by the author: the occurrence of jump points which are independent and make the process non-linear. The alternative to the latter feature would be some fairly continuous process when the estimation procedure would be similar to that of section 6 of the paper. A group of six other problems is mentioned where the analysis of the data is connected in a general way with the present paper. Four of these problems involved discriminating between non-overlapping composite hypotheses: a plausible criterion for this discrimination is by an extension of the Neyman-Pearson likelihood ratio. An example is given in connection with deciding whether a normal or log-normal distribution is the better " fit " to a random sample of observations.

Jessop commented that the use of a system of parabolic control lines in practice demanded the presence of a statistician and some adequately high grade technical staff on the production side. This is the price to be paid for making classical quantity control chart practice able to deal with situations where the underlying assumptions are not always satisfied, or what changes are desirable when a process is " out of control ".

Jowett described a medical application relating to records of growth and Stevens examined the problem from the point of view of closed-loop servo theory. Taking-up the author's mention of possible application to economic situations, Winsten commented that in economics the problem would most likely consist of trying to decide which of the possible causes acting at the times a jump might have occurred had an influence strong enough to produce the change.

Jenkins gave a general representation of the model used by the author in terms of " signal " and " noise " and noted that the author had confined his analysis to smoothing the " noise ". Some comments are given about the Bode-Shannon technique where spectral densities are used instead of autocorrelation functions:

BARNARD, G. A. (Report of discussion on a paper by)
Control charts and stochastic processes—*In English*
J. R. Statist. Soc. B (1959) **21**, 257-271 (1 figure)

8.9 (10.2)
continued

this linked with the servo mechanism discussed by Stevens.

Durbin discussed the relationship of the author's paper to other recent work in the field of econometrics relating to dynamic decisions taken under uncertainty. In particular, for the special case of a quadratic loss function, reference was made to papers by Simon [*Econometrica* (1956) **24**, 74-81] and to Theil [*Econometrica* (1957) **25**, 346-349]. It was also noted that in the formation adopted by the author an unbiased estimate (Y_t) of the direct variable (y_t) was observed. In this case the Theil-Simon solution required adjusting by a factor depending on the variance of the difference $(Y_t - y_t)$: the speaker had recently worked in this field.

Ewan contributed practical experience with a technique analogous to cumulative sum charts and said that it had been found possible to overcome the difficulty of the curve running off the graph paper. Other comments were given in connection with distribution of average run length and the fact that the methods used are applicable to range charts.

Moore offered some comments on the Average Run Length approach, the economic problem of whether to take large samples infrequently or smaller samples more frequently and the validity of the proposal that the cost of departures from the target value of a process is proportionate to the squared error.

Eisenhart discussed the so-called problem of the " premature plateau " in connection with the use of the progressive or cumulative average of observations. The disadvantage was that it could result in a premature stoppage of the observations. An " untidy " proof of this was achieved but it needed a rigorous proof and also relating to this present paper.

The written contribution by Page offered some general comment on the practice of cumulative sum charts—in particular the advantages of these over the ordinary statistical control charts.

(W. R. Buckland)

DURBIN, J. (London School of Economics) **8.1** (4.9)

Sampling theory for estimates based on fewer individuals than the number selected—*In English*

Bull. Int. Statist. Inst. (1958) **36**, III 113-119 (2 references)

The author draws attention to a surprising gap in the theory for estimating means and variances of proportions in finite populations as given in the majority of text books on sample surveys. This occurs when the number of individuals on which the estimate is based varies in an uncontrolled manner where sub-populations under study do not coincide with the stratification of the sample. In these circumstances the standard theory, which is based on fixed sample numbers, breaks down and must be replaced by one based on numbers which are random variables.

This situation had previously been discussed only in Yates' book [" Sampling Methods for Censuses and Surveys "] and no proof then accompanied the discussion, nor was it given much emphasis.

This present paper sets out to repair the omission by proving Yates' result and extending it to cover variable sampling fractions in two-stage sampling. The true variances as well as the variance estimates are quoted for the two sampling stages and the applicability of the method to more complicated situations is demonstrated.

(D. J. Page)

DURBIN, J. (Report of discussion on a paper by) **8.1** (4.9)

Sampling theory for estimates based on fewer individuals than the number selected—*In English*

Bull. Int. Statist. Inst. (1959) **36**, I 77-79

The contributors to this discussion were Cochran, Cornfield, Hansen and Healy with a brief reply by the author.

Cochran viewed the paper in relation to the problem of making analytical comparisons within a complex survey—a somewhat neglected branch of survey theory. A difficult example occurred when a multiple regression was computed from data within a multi-stage survey: the ordinary theory of regression did not apply. In connection with stratified samples, the use of supplementary information (if available) could sometimes avoid the increased variance referred to by the author in section 2 of his paper. With regard to the effect of non-response on the variance of the estimates, the assumption that the non-respondents were a sample from the same population as the respondents was frequently not realised in practice. Hence, some account should be taken of the bias arising from this non-response.

Cornfield raised a problem, suggested by the author's paper, concerning stratified populations. If n_{ij} is the sample in the jth stratum of the ith sub-population, then the effect of the sample-to-sample variations in the proportions $r_{ij}/\Sigma n_{ij}$ (where r out of n have a particular characteristic) on the confidence in the estimate $\Sigma r_{ij}/\Sigma n_{ij}$ could not be disregarded. A variance formula to reflect this would be desirable.

Hansen emphasised that an application of the theory for ratio estimation with stratified sampling—as given in standard sources—would give results identical with those of the author. However, the paper was an explicit demonstration of gains that might be achieved by following a particular plan of stratification.

Healy made the point that the amount of effort which could be diverted into the estimation of sampling errors was an important economic aspect of survey design and worthy of fuller study in both its practical and theoretical aspects.

(W. R. Buckland)

HÁJEK, J. (Math. Inst., Czechoslovak Academy of Science, Prague) **8.0** (4.9)
Some contributions to the theory of probability sampling—*In English*
Bull. Int. Statist. Inst. (1958) **36**, III 127-134 (6 references)

The paper consists of six short notes on different topics:

(i) *Confidence interval for the ratio of two means*

Fieller's method is compared with the usual " large sample approximation " for finding confidence intervals for $\mathscr{E}(y)/\mathscr{E}(x)$ when (x_i, y_i) are n independent and identically distributed pairs of observations. In this common situation it is found that Fieller's method is inferior.

(ii) *Upper estimate of sampling error in multi-stage sampling*

In multi-stage sampling the best estimate of variance for the estimated parameter cannot easily be obtained. It is simpler to find the upper bound of the variance and an unbiassed expression for this is derived, with an example.

(iii) *Poisson's sampling*

A procedure is described for sampling without replacement in the situation where the sample size may be permitted to vary but where a guarantee is required of the respective probabilities of inclusion of the various primary sampling units. This procedure, which has easily calculable and estimable variance, is designated " Poisson's sampling ".

The author describes an extension to two-stage sampling, where the first stage is random and the second stage Poissonian. He also gives a useful practical hint on using the Poisson process for sampling only one or a few elements with probabilities proportionate to given numbers $a_1, \ldots, a_n$, which may be used, *inter alia*, for obtaining an unbiased ratio estimate $S(y)/S(x)$.

(iv) *Optimum sampling strategy*

A technique is given for sampling on the basis of minimising costs for a given variance, or vice-versa, when the cost and variance functions are canonical in form.

(v) *The number of degrees of freedom of Student's* t *for finite population*

A defence is given of the use of $n-1$ degrees of freedom for the distribution of t when sampling without replacement from finite populations.

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HÁJEK, J. (Math. Inst., Czechoslovak Academy of Science, Prague) **8.0** (4.9)
Some contributions to the theory of probability sampling—*In English*
Bull. Int. Statist. Inst. (1958) **36**, III 127-134 (6 references)
continued

(vi) *Definitions of fundamental concepts*

Formal definitions are given, using the notation of sets, for the following terms:

population (universe);
probability sample;
estimate (statistic).

Sampling from a finite population is distinguished by the real existence and identifiabiltiy of its elements, rather than by the finiteness of the population. The arguments against defining a probability sample as a sequence are laid down.

(D. J. Page)

LAHIRI, D. B. (Indian Statistical Institute, Calcutta)

8.5 (4.9)

Observations on the use of interpenetrating samples in India—*In English*

Bull. Int. Statist. Inst. (1958) **36**, III 144-157 (10 references)

This paper is concerned with the technique of interpenetrating samples, introduced by P. C. Mahalanobis in 1937, which has been used with success for many years by the Indian Statistical Institute.

Confining himself to the problem of estimating population aggregates (or averages) from surveys, the author examines the efficiency of the method, under practical survey conditions, for measuring and controlling non-sampling errors. He classifies these errors under three headings:

(i) Coverage errors, (omission, duplication or wrong inclusion of elements.)
(ii) Errors in the recorded values of the characteristic under investigation.
(iii) Aggregation errors introduced during the process of averaging and weighting-up.

The test is one of reproducibility; abnormal errors being likely "by their very nature" to affect different interpenetrating samples very unequally. Although there are no obvious rules for deciding whether or not discrepancies between sets of calculations from interpenetration samples are due to random errors, habitual users of interpenetrating sample comparison methods develop a flair for discovering sources of error, some of which may have been quite unforeseen.

For such checks to be fully effective, interpenetrating sample comparisons must be sufficiently sensitive at "lower" levels, for example to compare two investigators, and suitably linked samples are preferable for such purposes to those which are (statistically) independent. But in either case the above method, by allowing the examination of many separate instances of apparently discrepant cases, can eventually lead to the discovery of a single clear instance and hence to the corrective measures necessary to remove the source of error.

When the number of interpenetrating samples is, say, five or more, and separate estimates based on each are available, then it is possible to use them directly without, any further calculations, for setting fairly efficient confidence limits to the mean of the parent population,

LAHIRI, D. B. (Indian Statistical Institute, Calcutta)

8.5 (4.9) *continued*

Observations on the use of interpenetrating samples in India—*In English*

Bull. Int. Statist. Inst. (1958) **36**, III 144-157 (10 references)

assuming that abnormal errors have already been virtually eliminated.

The benefits of this method can only be obtained by increasing survey costs, both from extra travelling time and from more laborious statistical analyses of the results. Experience in India suggests that the increase in expense is in many cases less than ten per cent.

The author concludes by listing two criteria on which to base the decision whether or not to adopt the interpenetrating sample method for a particular survey:

(i) How far the extra cost is justified in the light of expected benefits from measuring and reducing non-sampling errors.
(ii) How far it is possible to obtain equivalent advantages from any cheaper method.

On the former, he believes that there are few if any survey projects for which experiments would find non-sampling errors to be low enough to outweigh the extra cost of the method: and on the latter, he points out that experience in India (and elsewhere) has demonstrated the great advantages of the method over any alternatives which might, before comparison in operation, have appeared to provide equivalent improvements more cheaply.

(H. Palca)

KOLLER, S. (Federal Statistical Office, Wiesbaden) **8.7** (8.8)

On the problems of replicated sampling in German governmental statistics—*In English*

Bull. Int. Statist. Inst. (1958) **36**, III 135-143 (1 table, 1 figure)

The paper deals with the practical rather than the theoretical aspect of the subject of replicated sampling.

Primarily the paper deals, briefly, with replications in sub-samples. These may be pure replications, introduced to provide estimates of sampling error, or they may be deliberately arranged to provide information about certain factors; for example, interviewer bias. Then, at greater lengths, replications of the whole sampling procedure are discussed. Independent replications give different estimates of the characteristics to be examined. The paper deals with a method which should increase the chance of choosing a reliable sample.

Suppose the population value Y_0 of a characteristic is known from a previously conducted census, and it is desired to estimate the population value X_0 of some other characteristic. Different samples give different estimates of X_0 and Y_0, and the paper recommends that a sample be selected in which the estimated value of Y, $\hat{Y}$, is near to the known true value. If $\hat{Y} \neq Y_0$, and if Y and X are correlated a biased estimate of X_0 will be obtained. Alternatively, if a sample must be used in which $\hat{Y}$ differs substantially from Y_0, it is suggested that an improved estimate of X is obtained by taking $Y_{0\hat{X}\hat{Y}}$.

In practice the scheme is usually more complicated than this, because a larger number of characteristics is involved. An " acceptance inspection plan for samples " has been developed, in which all the previously measured characteristics in the selected sample are required to be within certain pre-determined limits of the true value. This technique has been applied chiefly in two-stage surveys such as the German Micro-Census, to ensure that a representative sample is obtained at the first stage.

(E. Weekley)

ODAKA, K. & NISIHIRA, S. (University of Tokyo and Inst. Statist. Math.) **8.1** (–.–)

Some factors related to social mobility in Japan—*In English*

Ann. Inst. Statist. Math., Tokyo (1959) **10**, 283-288 (6 references, 9 tables)

This paper summarises some findings on factors concerning social mobility, based on the data obtained from the Social Stratification and Social Mobility Survey conducted in 1955 by the Research Committee of the Japan Sociological Society. For the survey, appropriate sampling methods were considered. In order to cover not only a wide geographical area but also a wide range of subject-matter, one national survey and, in addition to this, five detailed surveys of four selected communities were made. The universe of the national survey was all Japanese males of ages 20～69, and 4500 individuals were chosen from electoral registers by the method of three-stage stratified random sampling. In each case of the detailed surveys, stratified random samples were also drawn from electoral registers.

(T. Suzuki)

LIEBERMAN, G. J. & BOWKER, A. H. (Stanford University, California)
Recent developments in continuous sampling—*In English*
Bull. Int. Statist. Inst. (1958) **36**, III 553-562 (16 references)

The purpose of the paper is to review the recent developments in the field of continuous sampling. These plans are used where batch-by-batch acceptance is impractical. The inspection is carried out by alternate sequences of consecutive item inspection and sequences of production which are either not inspected or from which sample items are taken. In all the plans discussed, items are classified as defective or non-defective.

The plan proposed by Dodge [*Ann. Math. Statist.* (1943) **14**, 264-279] may require abrupt changes from partial to 100 per cent. inspection and hence lead to administrative difficulties. To make this transition more gradual, two modifications of the original plan were made by Dodge & Torrey [*Industr. Qual. Contr.* (1951) **7**, 7-12].

Other plans which smooth the transition from sampling to 100 per cent. inspection are termed multi-level plans. Consecutive units are examined until i successive units are found without defects. Only a fraction f of the units are then inspected and should the next i items be non-defective, proceed to sample at a rate f^2 and so on. All these plans require one item to be selected from a block of $1/f^j$ items and the disadvantages of this are pointed out.

An alternative scheme is where each item has a probability f of being inspected and $i-f$ of not being inspected. There is a possibility of passing too many items without inspection and methods of truncating the sampling have been discussed.

Plans specifically designed to prescribe an average outgoing quality level have been obtained by Wald & Wolfowitz [*Ann. Math. Statist.* (1945) **16**, 30-49] and are forms of one-sided sequential plans. Another type of sequential plan is described by Girshick [*Technical Report No.* 16, Applied Mathematics and Statistics Laboratory, Stanford University, 1954]. The plans are explained for the general case, with the advantages and disadvantages outlined.

A criticism of all these continuous sampling plans is that they emphasise performing enough inspection to bring quality to the average outgoing quality level but do not provide automatic penalties for poor quality.

LIEBERMAN, G. J. & BOWKER, A. H. (Stanford University, California)
Recent developments in continuous sampling—*In English*
Bull. Int. Statist. Inst. (1958) **36**, III 553-562 (16 references)

The authors conclude by reviewing plans which provide for the termination of production until a source of difficulty has been located and removed. Several different mathematical models are used to represent a production process.

(M. J. Scott)

ROBSON, D. S. (Cornell University, Ithaca, N.Y.)
An unbiased sampling and estimation procedure for creel censuses of fishermen—*In English*
Biometrics (1960) **16**, 261-277 (4 references, 2 figures)

This paper is concerned with a creel census, estimation of the total number of fish removed over an entire fishing area. The author proposes a sampling procedure for obtaining an unbiased ratio estimate of the total catch which is the ratio of the estimated rate of total number of fish removed per fishing hour to the estimated total effort.

The area of the fishery is divided into segments in such a way that there is approximately an equal amount of fishing in each region. The fishing season is stratified into weeks and each week is stratified into three periods: Saturday, Sunday and the weekday period. Holiday weekdays are treated as separate strata. On each of d randomly chosen days in a weekday period, n area segments are selected independently without replacement. An enumerator in each region takes the day's count of the fishermen and the number of fish caught by each man. An additional enumerator, starting at a randomly selected place and time on each of the d selected days counts the number of fishermen in the entire fishery. The author then derives a formula for an unbiased estimate of the total fishing effort X for the entire season. In the same part there is a discussion on sampling errors in the unbiased estimate $\hat{X}$ and the error variance of $\hat{X}$. Next, the author deals with estimation of the total catch, Y. As in the previous part, he discussed the unbiased estimate of Y, and the corresponding sampling error variance of the estimate $\hat{Y}$.

In the last part of the paper, the following disadvantages of the sampling design are pointed out:

(i) The computational procedure is complicated.
(ii) Estimation of the error variance requires at least four census enumerators to work 2 or more days and an additional enumerator to make 2 or more trips on each of the randomly selected census days.
(iii) The size of a region which an enumerator can cover is limited.
(iv) The plan is impracticable when fishermen move from one region to another. Field personnel would not be used efficiently in segments where fishermen enter sporadically.

ROBSON, D. S. (Cornell University, Ithaca, N.Y.)
An unbiased sampling and estimation procedure for creel censuses of fishermen—*In English*
Biometrics (1960) **16**, 261-277 (4 references, 2 figures)

However, the method of estimation is more statisfactory when this field procedure is used than when enumerators move about freely, counting as many fishermen as possible. Efficiency of operation in the field is sacrificed for efficiency in estimation. Among the advantages of this method are freedom from statistical and subjective bias and the availability of estimates of the components of error variance.

(B. J. Trawinski)

SCHWARZ, H. (Hochsch. f. Ökonomie, Berlin) **8.1** (2.6)

Estimation of variance in planning sample surveys—*In German*

Statist. Praxis (1960) **15**, 111-114 (1 table, 7 figures)

The number of sample units needed in planning a simple random sample depends first on the precision required for the desired result at a determined significance level and secondly on the variance. If statistical information about the distribution is not available and performance of a pilot study is impossible, a rough estimate of the variance is obtained by using realistic assumptions about the range and shape of the distribution. In this paper the range (u, k) and the mean $\bar{x}$ of the distribution are assumed to be known approximately.

A first rough estimation of the variance is obtained by assuming a two-point-distribution, concentrated at the points u and k. This distribution is uniquely determined by the location of u, k and $\bar{x}$. The corresponding variance $s_0^2 = (k-\bar{x}).(\bar{x}-u)$ takes its maximum value at $\bar{x} = (u+k)/2$, that is if the mean is midway between u and k. A reduction of the variance estimate to one third may be obtained by assuming a double rectangular distribution, the probability density of which is determined again by u, k and $\bar{x}$.

More precise estimations of the variance, especially for the purpose of planning stratified sample surveys, are given by Schäffer & Nourney [*Metrika* (1958) **1**, 41-56] with the additional assumption of a linear or parabolic or exponential distribution.

(M. Nourney)

STANGE, K. (Technische Universität, Berlin) **8.8** (11.2)

Constructing range-plans for variables inspection by use of the double-probability-paper—*In German*

Metrika (1960) **3**, 151-165 (1 table, 4 figures)

The sampling plans for variables inspection (measurement) usually employ the standard deviation s. For a sample of size n the average $\bar{x}$ and variance s^2 is computed. On the basis of the test statistic $z = \bar{x}+ks$ one can decide upon the acceptance of the total lot. (If z exceeds a certain tolerance limit T, the total lot will be rejected; if $z \leqq T$, it will be accepted.)

The present paper describes a possibility for simplifying the use of variables in acceptance sampling. Instead of the standard deviation s of the total sample of size n, the mean range $\bar{R}$ of l sub-samples consisting of m pieces each $(n = l \times m)$ is employed to calculate the test statistic $z = \bar{x}+K.\bar{R}$. The characteristic magnitudes of a range plan are K and n. By use of the double probability paper, the operating characteristic curve of the range plan is approximately transformed into a straight line, provided that $l \geqq 3$ and $m \geqq 5$, so that it can be easily determined by two points, usually the consumer's and producer's risk. From the operating characteristic curve in the double probability paper the magnitudes K and n can be determined without calculation. By this method the user will become independent of the tabulated sampling plans and is put in the position of being able to prepare with little effort his own range plan for any given conditions.

Moreover, the author solves several other problems, such as how, with a given n and k, the overall sample has to be divided up in order to produce as steep an operating characteristic curve as possible. It is shown, that by the use of an R-plan instead of an s-plan the same power can be achieved by increasing the sample size by about 30 per cent.

The paper amplifies a previously published paper by the same author [*Metrika* (1958) **1**, 111-129; abstracted in this journal, No. 106, **8.8**].

(H. Schauerte)

Remark of the Regional Editor:

A series of three papers discussing the *practical application* of this method was published by the same author in "Qualitätskontrolle und Operational Research" (1960), 1-7, 29-34 and 75-80.

(J. Pfanzagl)

TALACKO, J. (Marquette University, Milwaukee, U.S.A.) **8.1** (2.5)

Special methods of inventory by sampling if the population sets have approximately negative exponential distribution—*In English*

Trab. Estadíst. (1959) **10**, 19-29 (16 references, 1 table, 1 figure)

Sets of inventories with strongly asymmetric distributions, and which suggest the hypothesis of a negative exponential distribution, are considered in this paper. In this situation rules are given for:

(i) Finding the best stratification.
(ii) For the best estimation of $N\bar{x}$, or of $\bar{x}$, from the grouped data.
(iii) Estimations of the variances of the population in every stratum.
(iv) Maximal assignment of samples of a given extension.

It is clear that stratified random sampling is adequate for the situation. The rules given are the following:

(i) The non-overlapping sub-populations, which are called " stratum ", are chosen from the populations with means in the interval $0{\cdot}5 \leqq \bar{x} \leqq 20$, in such way that the length of the intervals which form the strata are powers of some number a; that is the length of the i-interval is $a^i = a_i = x_{i+1} - x_i$, where $x_1, x_2, \ldots$ are the limits of the intervals.
(ii) If the data are grouped in unequal intervals, the argument X_i, in all the intervals except the first is supposed to have triangular distribution; for the first, a rectangular distribution is admitted.
(iii) It is possible to determine the variance of the population in each stratum without using the sampling values. The $\sigma_0^2 = a_0/2$, according to the rectangular distribution, and $\sigma_i^2 = a_i^2/18$.
(iv) In this situation the most convenient sampling method is the stratified one, with allocation proportional to the numbers $N_i\hat{\sigma}$, N_i being the number of elements of the i-stratum; these numbers are known.

In this way it is possible to obtain unbiased estimations of the means of the strata and of the population; the variance of the last one being the least possible. Estimations of the total of the population are found and

TALACKO, J. (Marquette University, Milwaukee, U.S.A.) **8.1** (2.5) *continued*

Special methods of inventory by sampling if the population sets have approximately negative exponential distribution—*In English*

Trab. Estadíst. (1959) **10**, 19-29 (16 references, 1 table, 1 figure)

confidential intervals of the mean and the total, with normal distributions for the classical estimators, are determined.

An example is given and the results are compared with the complete inventory.

(J. Béjar)

WEARDEN, B. L. van der (Zurich University) **8.8** (8.7)

Sampling inspection as a minimum loss problem—*In English*

Ann. Math. Statist. (1960) **31**, 369-384 (10 references, 5 figures)

The determination of economic sampling inspection plans for the producer or buyer of discrete lots of items is here approached through the losses incurred in sampling and in acceptance or rejection of lots having a proportion p of defectives. The sampling cost is taken proportional to n the sample size. The cost of accepting a lot of defective fraction p is taken to be ap, and that of rejecting this lot is taken to be $bp+c$, unless rejected lots are discarded, in which case " $b = 0$ ". Here a and b are known constants, $a>b$; and c is " the cost of inspection of the whole lot ".

For a given sampling plan, define $L(p)$ to be the expected loss incurred. The minimax approach to $L(p)$ gives a trivial answer. However, if the regret R is defined as $L(p) = L_m(p)$, where $L_m(p) = \min(ap+fn, bp+c+fn)$ is the minimum possible loss, given p with $bp+c$ replaced by $bp+c+fn$ for simplicity, then one can use the minimax approach on the regret. The author does this for two cases: (i) when p_0n and $(1-p_0)n$ are both " large ", in which case the normal approximation is used; and (ii) when n is large but p_0n is not, in which case the Poisson approximation is appropriate. Here p_0 denotes that " critical value" for which $ap_0 = bp_0+c$, i.e. where we are indifferent as to acceptance or rejection of the lot. The author states that these two cases were treated independently by Moriguti and by Ura, respectively [*Reports of Statistical Application Research, Union of Japanese Scientists and Engineers*, (1955) **3**, 99-121 and 140-148]; less generally but (in the case of Ura's paper) more extensively. The Poisson case is rather more involved than the other.

The buyer's situation, under the author's assumptions, leads to the same problem with $b = 0$, whether one considers the loss per lot or the loss per non-defective unit. A short discussion of sequential tests includes the opinion that the minimax-regret test is a Wald probability ratio test.

Finally, the problem is phrased in terms of game theory, and the possibility of a coalition is investigated. The problem of minimising the sum of the losses of producer and buyer is shown to be of the same type as the producer's problem, so that the methods developed carry over without modification.

2/139 (J. A. Lechner)

WILKS, S. S. (Princeton University) **8.1** (4.1)

A two-stage scheme for sampling without replacement—*In English*

Bull. Int. Statist. Inst. (1960) **37**, II 241-248 (4 references)

The author discusses a two-stage sampling scheme. The populations chosen are those with many primary units and sampling is without replacement at both stages in which the sizes of sub-samples to be selected without replacement from the primary sampling units are multiples of a number m. The multiples themselves are random variables and have a multi-dimensional hypergeometric distribution. The scheme considered here is an extension of one considered by Sukhatme in which sampling at the first stage is done with replacement, and that at the second without replacement. Values of a set of such variables can be readily determined by random numbers.

The unweighted mean of the entire sample drawn by the two-stage process is an unbiased estimator of the population mean. The variance of this estimator is determined; together with an unbiased estimator for this variance. The optimum choice of m is determined so as to minimise the variance of the sample-mean for a fixed total cost for the survey.

(S. S. Wilks)

ŽALUDOVÁ, Mrs. A. H. (Heat Engineering Inst., Prague) **8.9** (4.5)

Statistical quality control of production processes using individual sample values—*In English*

Bull. Int. Statist. Inst. (1958) **36,** III 573-578 (6 references, 1 table, 3 figures)

The paper describes a method of control charting where all the individual sample values are plotted and examined in relation to two pairs of control limits. The production process is considered satisfactory if none of the sample values fall beyond the outer limits and not more than one value falls in each of the regions between inner and outer limits.

The control limits are set in from the upper and lower tolerance limits for production according to the following formulae:

Outer Limits
Upper $T_u - l_1 T$
Lower $T_L + l_1 T$
Inner Limits
Upper $T_u - l_2 T$
Lower $T_L + l_2 T$

Where T_u is upper tolerance limit
T_L is lower tolerance limit
$T = T_u - T_L$

The coefficients l_1 and l_2 are tabulated for sample sizes $n = 3, 4, \ldots, 10$ and for permissible percentage defective $P = 2, 1, 0{\cdot}5$ and $0{\cdot}27$; with risks $\alpha = 0{\cdot}05$ and $\alpha_1 = 0{\cdot}005$ of making the wrong decisions when using the control chart.

The usual assumptions concerning the performance of the process with regard to the tolerance and the assumption of normality are stated, then the computation of control limits is described in detail.

It is claimed that this type of control chart has advantage over the more orthodox methods in the simplicity of its operation whilst maintaining reasonable sensitivity in detecting changes in the production process.

(Pamela F. Slight)

BAILEY, N. T. J. (University of Oxford)

The use of linear algebra in deriving prime power factorial designs with confounding and fractional replications—*In English*

Sankhyā (1959) **21**, 345-355 (17 references)

This paper gives an account of the results of two previous papers [Rao, C. R. *Bull. Cal. Math. Soc.* (1946) **38**, 67-78; and Bose, R. C. *Sankhyā* (1947) **8**, 107-166.] and the author suggests some quick procedures for the construction of certain type of designs.

In p^n factorial designs with confounding and fractional replication, the treatment combinations comprising a block are derived from complete solutions of certain systems of linear equations each having the same left hand side. The author reduces the left hand side to its canonical form.

Designs with p prime, are first considered. Here the algebra of remainders modulo p provides a field. For designs with block size p^{n-m}, the reduction gives a system of m equations in which each of the m canonical variables have unit coefficients in one equation and zero in the rest. It is easier to check with this echelon system that no interactions of order less than a certain specified number are confounded. A basis for the solution of homogeneous equations can be readily constructed from the echelon system. The group having this basis as a set of generators gives the intra-block subgroup. For any other block, complete solutions are obtained by adding any particular solution obtained for the intra-block subgroup. For complete confounding, one treatment combination for every block is easily obtained by taking for the canonical variables all combinations of 0, 1, 2, ..., $p-1$ and putting zero for the rest. For fractional replications, with small fractions, the use of a matrix pre-multiplier, which transforms each set on the right hand side into a set of values for the canonical variables is suggested. This discussion is followed by an illustrative example of a 1/9th replicate of a 3^7 design in nine blocks.

A simple proof of Fisher's theorem on minimum block size based on this echelon system is given. Fisher's result regarding the derivation of a p^n design from a 2^n design is obtained. From a 2^n design in which m

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independent interactions are confounded, a p^n design, confounding m interactions, each involving no fewer interactions than in the original arrangement is also obtained.

The paper also discusses the case where the number of levels for each factor is prime power.

(K. R. Shah)

In spite of the apparent advantages of sequential medical trials, medical research workers seem to have a reluctance to adopt the technique on a wide scale. The author investigates some of the reasons for this under three headings: " psychological ", practical and economic.

Under the first heading he lists the implied finality of the trial as being unacceptable to the research worker who is looking for a guide rather than a directive although a comparative trial of two or more drugs is essentially an acceptance-rejection test. Other problems under this heading include the fact that the open ended trials commit the worker to an unlimited amount of work unless a truncation is applied. Furthermore a clinical trial will often involve the simultaneous study of a number of characteristics and if each is to be separately followed sequentially the operation of the trial becomes cumbersome.

The practical difficulties are not, in general, greater for sequential than non-sequential trials. In some cases the sequential method is not ideal—for example where the treatment is short but the assessment is a lengthy process. One drawback of the sequential system is in the handling of " defaulters ". There tend to be many of these in clinical trials, for a variety of reasons, and whereas in the " straight " trial each defaulter represents a loss of one unit the sequential defaulter results in the loss of a " patient-pair ".

In general there are obvious economic advantages in favour of the sequential trial. It is reiterated that there is no obvious reason why a truncating procedure can not be applied in determining an upper limit to the amount of work required. Anscombe [*Biometrics* (1954) **10**, 89] showed that classical methods can validly be applied to data at the stopping point provided this is predetermined.

The paper ends with a description of a trial organised by the unit to determine the effect of weight regulation on the incidence and severity of the pre-eclampsia of pregnancy. Four matching criteria were selected and the limits within which experimental and control cases must agree were decided for each criterion. This method of " moving grid " matching was preferred to matching within rigidly defined strata although the author later

discloses that two sets of control patients were used, one determined on the moving grid principle and the other randomly selected. So far the two series have behaved similarly.

The diagnostic criteria allow easy assessment of severe pre-eclampsia but definition of mild cases presented the usual difficulties. The author therefore compromised by selecting all hypertensive complications, severe pre-eclampsia and incidence of prematurity as three independent characteristics. This has had the interesting result that, so far, a non-significant result has been reached for all hypertensive complications at about the same point in time as a significant result for severe pre-eclampsia.

(J. J. Grimshaw)

BOX, G. E. P. (Princeton University) **9.4** (9.3)

Use of statistical methods in the elucidation of basic mechanisms—*In English*

Bull. Int. Statist. Inst. (1958) **36**, III 215-224 (9 references, 1 table)

The author distinguishes three phases in exploring a functional relationship of the kind:

$$\eta = f(\xi_1, \xi_2, \ldots, \xi_n;\ \theta_1, \theta_2, \ldots, \theta_n) = f(\xi;\ \theta)$$

(i) Screening studies to delineate important variables;
(ii) Empirical studies to describe certain important features of the relationship between the response and the variables;
(iii) Theoretical studies aimed at discovering predicting mechanisms for the phenomena studied and estimating the relevant parameters.

The problems of design and analysis which arose in the third phase are studied in this paper. An example is given of a two-stage chemical decomposition: $A \xrightarrow{k_1} B \xrightarrow{k_2} C$ where k_1 and k_2 are the rate constants for the two reactions.

In this particular example, although not in general, the differential equations governing the yields of A, B and C give an explicit solution. The problems of analysis are to discover whether the assumed functional form is adequate to describe the data and to estimate values of the unknown parameters as well as to determine their precision. A method for finding the least-squares estimates of the constants, as originally described by Gauss, is presented and the data given are processed by this method as a comparison with the results of processing the same data by the method of " steepest ascent " [Box & Coutie, *Proc. Instn. Elect. Engnrs.*, B (1956) **103**, Suppl. 1]. An approximate confidence interval is derived for the θ's.

Some problems also arise on the experiment design. One arises where two or more mechanisms are equally plausible on present data and we have to design additonal discriminating experiments. Another problem occurs where the original data are unsuitable for estimating separately the constants θ and we wish to generate data to allow the best possible estimates to be obtained.

It is sometimes found that some estimates of the θ are very highly correlated. The sums-of-squares surface then involves valleys near minima and a number of the latent roots (r) of the correlation matrix approach

BOX, G. E. P. (Princeton University) **9.4** (9.3) *continued*

Use of statistical methods in the elucidation of basic mechanisms—*In English*

Bull. Int. Statist. Inst. (1958) **36**, III 215-224 (9 references, 1 table)

zero. This indicates that only $p-r$ linear combinations of the parameters can effectively be estimated separately.

The author describes a method of approach leading to the criteria for selection of the optimum design. He emphasises that, if design is possible at all, some form of preliminary estimates will be available. An assumption is made that the number of distinct trials is equal to the number of parameters to be estimated. Ordinarily, however, there is a two-fold objective, (i) to check the model, (ii) to estimate the parameters in it when it appears to be adequate. It is shown that if an alternative wider model can be specified, both the above objectives may be achieved. This is exemplified in terms of the chemical decomposition problem discussed earlier.

(J. J. Grimshaw)

DAVIES. O. L. (Imperial Chemical Industries, Macclesfield, England) **9.4** (8.8)
The design of screening tests in the pharmaceutical industry—*In English*
Bull. Int. Statist. Inst. (1958) **36**, III 226-240 (4 references, 1 table, 2 figures)

This paper is concerned with the selection of the optimum test procedure applicable to a given screening problem. The introduction gives a description of two basic types of screening test exemplified by chemotherapy of tuberculosis, and screening of mutants in antibiotic research.

The author draws an analogy between the statistical characteristics of screening tests and those of acceptance sampling schemes. As with the latter, screening tests may involve a single stage, two stage or sequential procedure and, in general, the sequential process will be the most economic where it is applicable.

Three criteria can be used to define " optimum procedure ":

(i) Reduction of the number of compounds to a given fraction so as to maximise the *number* of active compounds per unit of effort.
(ii) Reduction to a given fraction so as to maximise the mean activity per unit of effort.
(iii) Specification of proportion of positives to false positives emerging from the test and the design of a test to maximise the number of active compounds per unit of effort, accordingly.

The author considers the first and last of these criteria to be the most useful in biological screening and after a description of the economics of optimum sampling he selects for maximisation the ratio P/C, where P = No. of positive compounds emerging from the screen and C = average operating cost per compound tested.

The mathematics of the sequential test procedure are briefly described and a series of curves is given from which the average number of tests required by a negative and also the proportion of positives emerging from the test can be calculated.

The two-stage test is described in some detail. The author stresses the frequently observed fact that for all practical purposes compounds submitted to a screening programme can be regarded as being dichotomised into a very large number with zero activity and a few active ones of activity a (the selection barrier level). Using the tuberculosis screening procedure as an example he gives

DAVIES, O. L. (Imperial Chemical Industries, Macclesfield, England) **9.4** (8.8)
The design of screening tests in the pharmaceutical industry—*In English*
Bull. Int. Statist. Inst. (1958) **36**, III 226-240 (4 references, 1 table, 2 figures)
continued

a useful table from which an optimum design can be chosen for this type of test.

It is pointed out that the procedures described are all based on the principle of random selection of unknown compounds. When " following a lead " considerably more positives would be expected to arise, the selection of compounds is no longer random and different considerations apply.

(J. J. Grimshaw)

DAVIES, O. L. (Report of discussion on a paper by) **9.4** (8.8)
The design of screening tests in the pharmaceutical industry—*In English*
Bull. Int. Statist. Inst. (1959) **36**, I 85-86

Those taking part in the discussion were Finney and Dunnett: the author briefly replied.

Finney said the approach by the author was closely related to his own and both emphasised the warning against overconfidence in trying to select a winning item from a large number of alternatives. Well-planned two- or three-stage selection would generally achieve most of what was economically practical. The comparative intractability of the mathematical theory beyond two stages should not be allowed to exert undue influence. He suggested that a view of the external economy could be of real value in modifying the balance between effort in testing new compounds compared with that involved in their production. This could be of importance to the development of the statistical methods used in this field.

Dunnett suggested approaching the design of screening tests as a decision problem where the unknown state of venture was the true activity of the compound. The cost of wrong decisions would need to be evaluated and some assumption made about the *a priori* distribution of activity in order to set up a Bayes' procedure.

(W. R. Buckland)

DYKSTRA, O. (General Foods Res. Center, Tarrytown, N.Y.) **9.5** (9.1)
Rank analysis of incomplete block designs: a method of paired comparisons employing unequal repetitions on pairs—*In English*
Biometrics (1960) **16**, 176-188 (8 references, 7 tables)

In their approach to the analysis of a paired-comparison experiment involving t treatments with n repetitions Bradley & Terry [*Biometrika* (1952) **39**, 324-345] postulated that associated with each of the treatments, denoted by $T_1, \ldots, T_t$, there exist parameters π_i for T_i such that $\pi_i \geqq 0$ and $\sum_{i=1}^{t} \pi_i = 1$. The parameters are further defined by the probability statement that, if X_i denotes an observation on a sample of T_i,

$$\Pr(X_i > X_j) = \pi_i/(\pi_i+\pi_j)$$

in the comparison of T_i with T_j, $i \neq j$.

The author considers the same model in the case of unequal repetitions on pairs. He generalises results of Bradley & Terry:

(i) to obtain maximum likelihood estimates p_i of the π_i;
(ii) to test the null hypothesis $\pi_i = 1/t$, all i;
(iii) to test the appropriateness of the model.

Large-sample formulae for the precision of the estimates p_i are then obtained and used to give an approximate expression for the variance of $p_i/(p_i+p_j)$, the estimate of $\Pr(X_i > X_j)$. The author compares this approximate variance with that of the binomial proportion of preferences for T_i in N_{ij} direct comparisons of T_i and T_j. In the special case where all π_i and repetitions are equal the same precision results if $N_{ij} = \frac{1}{2}nt$.

A balanced paired-comparison experiment with n repetitions involves $\frac{1}{2}nt(t-1)$ comparisons. For large t it may be desirable not to make all possible comparisons equally often, even if the number of repetitions between any two pairs is at the choice of the experimenter. The author considers in detail the situation where this number is either n or zero. Suitable partially balanced incomplete block designs have been given by Clatworthy [*J. Res., Nat. Bur. Stand.* (1955) **54**, 177], and may be sub-classified into group-divisible, triangular, square and cyclic designs. Many of these designs are listed in the present paper together with their efficiency, this being defined as the average pair-wise precision divided by the precision when the same number of judgments is spread equally over all $\frac{1}{2}t(t-1)$ pairs.

A worked example illustrates the basic computational procedures.

(H. A. David)

FINNEY, D. J. (University of Aberdeen) **9.4** (9.7)

Statistical problems of plant selection—*In English*

Bull. Int. Statist. Inst. (1958) **36**, III 242-268 (11 references, 3 tables, 8 figures)

The author supposes that there are N new varieties of an annually sown and harvested crop which form part of a series of trials from which to select each year according to yield, and that the successive selections are made for k years, until only n varieties remain. The problem involved, of the best allocation of resources to increase the potential yield of the n varieties, is fundamentally a statistical one. If it be assumed that the yields of the N varieties are normally distributed, a solution for $k = 1$ is found without difficulty. A table and graphs are presented, to show the increase in average yields and its dependence upon n/N and upon the variance of experimental error. If n/N is small there is some benefit in discarding, at random, a number of the N varieties without testing them at all: this increases the precision of the trial for the remainder.

When k is 2 or more, consideration needs also to be given to the best allocation of the total available area amongst the trials corresponding to the k stages. Using a mathematical model (which should be an adequate guide for most sets of selections from finite numbers of varieties) the author has devised a method of calculating the consequences of each successive selection, but it is a major computing task to continue beyond $k = 2$. An analysis of the $k = 2$ case indicates that optimum results can be approximately reached if each year a fraction $(n/N)^{\frac{1}{2}}$ is chosen, and one-half of the total area is allocated to each stage. For this case and for $k = 2$ the advantage of rejecting some varieties at random without trial is usually negligible.

There are reasons for believing that a general rule, leading to results which are not optimum for all k, would be to share the available area equally between each stage and to use the successive yield results to choose a fraction $(n/N)^{1/k}$ of the varieties to be subjected to the next trial.

In a final section, the author studies in outline the way in which relevant economic factors determine the best choice of k, N, n, and the total available area. The question is raised as to whether really long selection programmes may not lose far more from time lag than they gain from additional improvement in yield.

(H. Palca)

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FINNEY, D. J. (Report of discussion on a paper by) **9.4** (9.7)

Statistical problems of plant selection—*In English*

Bull. Int. Statist. Inst. (1959) **36**, I 87-88

Those taking part in the discussion were Gumbel, Cochran, Fisher and Hotelling: the author briefly replied.

Gumbel suggested that, because the author had chosen at each stage the plant having the maximum yield (as measured on a continuous scale), the asymptotic theory of largest values was appropriate. Consequently there was a need to know something of the initial distribution: it had been shown that in some cases the average of the extremes was normally distributed. In the author's second case this was not so and the log-normal distribution was appropriate for the third.

Cochran drew attention to the plant-breeder's maxim of "throwing away": a poor breeder would need increasingly large amounts of land because of a low power to select and discard. With regard to the choice of a new variety, practice agreed that the problem was multivariate. As a suggestion, the F-ratio of the mean-square might be used as a guide to the intensity of selection; discarding more drastically when F was high.

Fisher drew attention to the case for quicker tests and rapid release of new varieties or products and Hotelling linked the author's paper with the recent work (and papers) on generalised sequential experimentation.

(W. R. Buckland)

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FINNEY, D. J. (University of Aberdeen)

A simple example of the external economy of varietal selection—*In English*

Bull. Int. Statist. Inst. (1960) **37**, III 91-106 (5 references, 5 tables, 1 figure)

In continuation of the theme of a previous paper [" Statistical problems of plant selection ", *Bull. Int. Statist. Inst.* (1958) **36**, III 242-260; abstracted in this present journal No. 154, **9.4**] an attempt has been made to examine the economically optimal balance between expenditure of effort on breeding and testing new varieties of a crop plant and the gain to agricultural production resulting from adoption of those selected as improvements on existing standards. Important factors involved are the average cost of breeding a new variety, the cost of conducting field trials, the statistical precision of these trials, the genetic variability between potential new varieties, and the value to the community of unit increase in the mean yield from the area under the crop.

Under the simple assumptions of the previous paper and for the special case of selection in one stage, the conditions for maximising the net gain, by optimal choice of the number of new varieties to be produced for testing annually and of the total area to be assigned to the field trials, can be expressed fairly easily in terms of the factors mentioned. The equations have been given and a group of numerical examples that might approximate to the situation in cereal breeding have been explored. These suggest that the optimal annual expenditure on breeding and yield-testing of varieties might be roughly at the rate of £1 for every £6·5 gain that would result from unit increase in yield on every acre under the crop. Numerical values used in estimating this rate were deliberately guessed conservatively: even allowing that the estimate may still be too great by a factor of three, it almost certainly corresponds to an expenditure vastly in excess of that currently allowed for cereals in Britain or elsewhere.

The approximation to actual selection programmes is crude, but a similar analysis for a more realistic scheme involving selection spread over two or more successive stages appears to be mathematically quite intractable. Reasons have been advanced, however, for believing that the optimal allocation of expenditure is unlikely to be less than it would be for one-stage selection and may be not very much greater. To judge how the net gain from multi-stage selection compares with that from

FINNEY, D. J. (University of Aberdeen)

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one-stage selection is less easy, but any figure obtained for one-stage must be a lower limit to the corresponding results for several stages, except when the cost of delay from the number of stages becomes important. In this paper, no account has been taken of this last factor, and no consideration has been given to the problem of estimating the optimal number of stages.

(D. J. Finney)

FINNEY, D. J. (Report of discussion on a paper by) **9.4** (9.7)
A simple example of the external economy of varietal selection—*In English*
Bull. Int. Statist. Inst. (1960) **37**, I 74-76

Those taking part in the discussion were Grimm, Quenouille, and Bradley: the author replied briefly.

Grimm referred to methods used in micro-biology which were similar to those outlined in the paper. However, in the case of micro-biology the interval from experiment to experiment was one of only a few weeks which made multi-stage and sequential planning preferable; one other difference was important—that in micro-biology the normal distribution was not always found in practice.

Quenouille raised a point in connection with the author's reference (in section 2) to an empirical relationship derived by Fairfield-Smith. If it was assumed that a circular Markoff process existed in the field, then the relationship was not wholly linear but tended to be simoid at the ends. This theoretical relationship, therefore, supported the empirical one in the centre part of the range of plot size.

Bradley drew attention to similarities between the problem discussed by the author and the work of Somerville entitled "Some problems of optimum sampling" [*Biometrika* (1954) **41**, 420-429].

(W. R. Buckland)

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EMMENS, C. W. (Department of Veterinary Physiology, Univ. Sydney) **9.0** (7.5)
The role of statistics in physiological research—*In English*
Biometrics (1960) **16**, 161-175 (9 references, 13 tables, 1 figure)

This paper presents the substance of a Presidential Address delivered at the Biennial Meeting of the Australasian Region of The Biometric Society in Melbourne, August, 1959.

Although physiologists have been accustomed to treat their measurement of responses mathematically in attempts at curve fitting, the statistical evaluation of the results or the planning of experiments along statistical lines is not frequently seen.

Many individuals have thought that animals are not very adaptable to statistical handling and that results of experiments of similar structure to those seen in agriculture would be at least as complex to handle, with frequent interactions and sometimes with difficulty in interpretation. The author shows that this is not the case. In fact factorial designs, Latin squares, and other partially replicated designs less often show significant interaction in physiology than in agriculture or industrial chemistry.

Most of the physiological research can be considered as falling into one or more of four main classes depending on whether comparisons are made (1) between or within animals, or (2) preparations from them; and on whether observations are (3) quantal or (4) graded. At worst, a graded response gives about twice the information that is given by a quantal response, therefore it is worth-while to convert from the quantal observation to a graded observation for both between and within animals comparisons.

Summaries of actual experiments utilising various types of designs are given for both graded and quantal responses where the comparisons are made both between animals and within animals. These summaries support the usefulness of designed experiments in physiological research and confirm the above statements.

Although all the computations of the summarised experiments were made on desk machines, the full development of statistics in physiological research will depend on the availability of automatic computers. This is due to the fact that automatic computers will allow the use of complex designs, both in planning and analysis and the computation of results will take only a fraction of the time required by desk machines.

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(C. Y. Kramer)

IRWIN, J. O. (MRC Statistical Unit, London School of Hygiene) **9.4** (9.1)

Experiments on the carcinogenic action of mineral oils—*In English*

Bull. Int. Statist. Inst. (1958) **36**, III 184-192 (5 references, 5 tables)

This paper describes some of the long series of collaborative tests being carried out to identify the fractions of mineral oils which are highly carcinogenic. Carcinogenic tests are laborious, taking fifty-two weeks to perform, and fractions tested in any one year are selected with reference to the previous year's biological results and also on the results of the chemists' fractionation experiments. As a result the tests, which commenced in 1950, are still continuing and are being carried out at Birmingham, Leeds and the Chester Beatty Institute in London.

Both mice and rabbits are used in the screening procedure. Rabbits are by far the more useful animals as they are more sensitive to the skin carcinogens of the type encountered and different fractions can be compared on the same animal. The mouse screen is continued to offset a possible species variation.

As a description of the type of work involved, the 1952-53 Birmingham series of tests is described in which twenty-one fractions were tested including dilutions each of two standard carcinogens: 9 : 10 dimethylbenzanthracene and methylcholanthrene.

In mice three measures of response are used: (i) Expectation of tumourless life—based on an actuarial model and truncated at fifty-two weeks, (ii) Date when ten per cent. of survivors had tumours, (iii) Total number of tumours. In the experiments described 1050 mice were used and the results are tabulated for each fraction under the three separate response metameters.

In rabbit experiments a Youden square design is applied. " Blocks " of five animals are used and the measure of response is the number of tumours per site out of the possible. The results are analysed by analysis of variance following an angular transformation.

The design used in the 1952-53 series of tests is tabulated in full and part of the variance analysis is given. The angular transformation actually used is

$$y = \sin^{-1}\sqrt{(r+3/8)/(n+\tfrac{3}{4})}$$

where r = number of positive responses out of n. It is

IRWIN, J. O. (MRC Statistical Unit, London School of Hygiene) **9.4** (9.1) *continued*

Experiments on the carcinogenic action of mineral oils—*In English*

Bull. Int. Statist. Inst. (1958) **36**, III 184-192 (5 references, 5 tables)

claimed that this removes bias in the mean and more nearly equalises variance at different levels than the more usual sine transformation $y = \sin^{-1}\sqrt{r/n}$.

The relative potencies of the two standards was about the same in mice and rabbits but some of the subfractions used were much less patent in mice, although the relative order of the fractions was approximately the same in both sets of experiments. Since it requires twenty rabbits, each painted on five sites, to achieve the same precision of estimate as fifty mice, the rabbit is the more useful because of its greater sensitivity.

The author points out that a tentative " dimethylbenzanthracene equivalent " can be given to any fraction although this must be treated with reserve as it assumes an equality of the dose-response slopes. However, using these equivalents it is possible to estimate the extent to which the carcinogens in an oil are being concentrated in particular sub-fractions of it.

(J. J. Grimshaw)

IRWIN, J. O. (Report of discussion on a paper by) **9.4** (9.1)
Experiments on the carcinogenic action of mineral oils—*In English*
Bull. Int. Statist. Inst. (1959) **36**, I 131-133

Those taking part in the discussion were Finney and Healy (presenting a written contribution by Miss Sandomire): the author replied.

Finney asked whether there could be any advantage in using different Youden squares as replicates of an expression instead of repeating the same squares. It thought that this might be an unnecessary complication but worth considering in the light of having different trials more nearly balanced as between treatments.

Miss Sandomire (read by Healy) questioned the appropriateness of the experimental criteria, particularly in relation to the length of the experiment. In the selection of oils for further study, use of the Scheffé method [*Biometrika* (1953) **40**, 87-104] for judging all contrasts in the analysis of variance might prove beneficial.

(W. R. Buckland)

JUSTESEN, S. H. & KEULS, M. (Agricultural Univ., Wageningen) **9.7** (–.–)
Notes on the use of non-orthogonal designs—*In English*
Bull. Int. Statist. Inst. (1958) **36**, III 269-276 (3 references, 4 tables)

Designs which have been suggested to overcome heterogeneity usually possess a certain symmetry, which simplifies the statistical treatment of the results. These designs are not sufficiently flexible for some people, for example, for plant breeders, who often have a large number of varieties with few replications. Designs suitable for this type of situation are more difficult to analyse, and the note recommends the use of the iterative method first described by Stevens [*Biometrika* (1948) **35**, 346-367]. The reasoning by vectors, due to Kuiper [*Statist. Neerlandica* (1952) **6**, 149-194] is employed.

The note first shows the application of this approach to the orthogonal case and demonstrates that it gives the same results as an analysis of variance. The authors then show how the non-orthogonal case may be treated using this technique. Because of the lack of orthogonality, the given iterative process must be used, the process being repeated until the required degree of precision is obtained.

In conclusion, the note recommends some principles to be observed in choosing a design, and gives two examples.

(E. Weekley)

KIEFER, J. (Cornell, Ithaca, N.Y. and Oxford Universities) **9.0** (9.4)
Optimum experimental designs—*In English*
J. R. Statist. Soc. B (1959) **21,** 273-304 (61 references)

This paper, which is primarily a summary of recent developments in its chosen field, is composed of two major parts. In the first part the author gives some simple examples to show how important it is that the aims of an investigation should be clearly understood when planning an experimental design for it. In the second part he gives a catalogue of various types of optimality criteria that might possibly be used in different situations, and discusses some relations between them. Although it is stated that " our approach will be in the spirit of Wald's decision theory " there is little consideration of the cost of sampling or of the cost of wrong decisions. A very general approach to the assessment of the relative importance of various errors is provided.

The following topics are considered in the first part of the paper:

(*a*) Randomisation and degrees of freedom.

Designs which are optimum for estimating parameters may not be optimum for testing hypotheses about parameter values, if they give estimators of the residual variance with relatively few degrees of freedom. As an example, the author considers a one-way classification by groups, with three groups, having unknown (fixed) means θ_1, θ_2 and θ_3; six observations are to be allocated to these three groups. What are the optimum allocations if the objective is

(i) point estimation of θ_1, θ_2 and θ_3?
(ii) testing the hypothesis $\theta_1 = \theta_2 = \theta_3 = 0$?
(iii) testing the hypothesis $\theta_1 = \theta_2 = \theta_3$?

It is shown that, for (i) two observations should be taken in each of the three groups; for (ii) a single group should be chosen at random and all observations taken from this group; for (iii) two groups should be chosen at random and three observations taken from each group. In cases (ii) and (iii) these designs are appropriate only if the departures from the stated hypothesis are " small " in magnitude.

(*b*) Use and mis-use of the *F*-test.

The *F*-test if optimal with respect to alternative hypotheses classified according to a certain

KIEFER, J. (Cornell, Ithaca, N.Y. and Oxford Universities) **9.0** (9.4)
Optimum experimental designs—*In English*
J. R. Statist. Soc. B (1959) **21**, 273-304 (61 references)
continued

function (the " non-centrality parameter " of the distribution of the mean-square ratio) of the parameters. It is pointed out that this function is not always the one appropriate to a particular problem; even when originally accidental effects—such as missing observations—may change the form of the relevant function.

(*c*) Discreteness.

The author gives, as example, the computational nuisances arising for certain values of sample size (with special reference to choice of optimal spacing for the independent variable in problems of polynomial regression). He suggests that approximately optimal designs, valid for all samples, might be used.

(*d*) Methods of allowing for hetero-scedasticity and variable cost, and

(*e*) Invariance.

For example, invariance of form of a regression function with respect to a group of transformations are also briefly discussed.

The second part of the paper commences with a general theoretical discussion, in the course of which the author's definitions of admissibility and complete classes are expounded. He claims that by using these concepts it will be possible to construct sets of designs which are good for a given problem. The statistician can search amongst these for a design to recommend.

This is followed by a catalogue of specific optimality criteria. It is supposed that we are interested exclusively in the values of s linearly independent linear functions of the parameters θ_i, $\psi_j = \sum_i c_{ij}\theta_i$, $(j = 1, 2, \ldots, s)$. A test procedure testing the hypothesis $\psi_1 = \psi_2 = \ldots = \psi_s = 0$ is denoted by ϕ; a design is represented by d; and V_d denotes the variance-covariance matrix of the best linear unbiased estimates of $\psi_1, \psi_2, \ldots, \psi_s$, when the design d is used. Let

$$\bar{\beta}_\phi(c) = \inf\left(\text{power function of } \phi \text{ when } \sum_1^s \psi_j^2/\sigma^2 = c\right)$$

and $\bar{\beta}(c, d, \alpha) = \sup_\phi \bar{\beta}_\phi(c)$ for all ϕ with significance level α based on design d.

KIEFER, J. (Cornell, Ithaca, N.Y. and Oxford Universities)
Optimum experimental designs—*In English*
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9.0 (9.4)
continued (2)

Then a design d^* is:

(i) $M_{\alpha, c}$-optimum if $\bar{\beta}(d^*, c, \alpha) = \max_d \bar{\beta}(d, c, \alpha)$,

(ii) L_α-optimum if

$$\lim_{c \to 0} [\{\bar{\beta}(d^*, c, \alpha) - \alpha\} / \{\sup_d \bar{\beta}(d, c, \alpha) - \alpha\}] = 1,$$

(iii) D-optimum if $\det v_{d^*} = \min \det v_d$,

(iv) E-optimum if d^* minimises the maximum eigenvalue of V_d

(v) A-optimum if d^* minimises the trace of V_d (i.e., minimises the sum of variances of estimators of $\psi_1, \psi_2, \ldots, \psi_s$.

Relationship between these criteria are discussed. For example, it is shown that for the standard balanced designs (Latin squares, randomised blocks, etc.) A-, D-, E- and L_α-optimality are closely related, although this is not so in the more general cases.

A further definition, G-optimality, is defined for regression problems. A G-optimum design minimises the supremum (with respect to the independent variable vector) of the variance of (predicted—true) values of the regression function.

(N. L. Johnson)

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KIEFER, J. (Report of discussion on a paper by)
Optimum experimental designs—*In English*
J. R. Statist. Soc. B (1959) **21**, 304-319 (1 table)

9.0 (9.4)

Those taking part in the discussion were Tocher, Quenouille, Mallows, R. N. Curnow, Yates, Beale, Cox, Barnard, R. C. Curnow and M. Stone. The author replied in detail.

Tocher said that the implications that experiments must have a precise objective so that the mathematicians can help them appeared to be unwarranted. What was required was a robust experiment that would give useful—if not the optimum amount of—information on a number of different questions. The author has modelled his theory of design on the theory of decision function and, just as that had been shown not to provide a complete theory of inference, so an attempt to find a design theory on such lines is insufficient. The future for the theory of experimental design is in a sequential theory providing that such a theory gives freedom of choice of new objectives after each stage.

Quenouille suggested that the theme of the author's paper was that there exists an optimum design for an experimental problem, and that this design depends upon the set-up and method of measurement as well as the problem itself. He, too, emphasised the point of robustness of design—both in terms of the range of questions that can be answered as well as performance in face of difficulties such as missing observations.

Mallows proposed that, while the author's paper forced careful consideration as to what was required from a design, there was nothing wrong with starting with an intuitively attractive design and searching for the appropriate optimal criteria. Some work of Box & Draper was referred to, dealing with the effect of presence of higher order terms when fitting a polynomial regression.

R. N. Curnow drew attention to a resemblance between the author's random selection of treatments for actual observations and the random selection of varieties in a varietal trial. Yates also referred to this aspect and the points raised by Tocher—particularly to consequences of " concentrating on alternatives close to the null-hypothesis " in relation to the power of an experiment to detect a difference.

Beale commented on the question of sequential design

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and referred to some unpublished work of his own which tried to give a procedure which could be alternative to the Box-Wilson procedure. Cox showed that the author had concentrated upon the problem of minimising the effect of random errors and proceeded to comment upon the aspect of experimental designs which concerned itself with the examination of interactions. Barnard referred to the origins of the Box-Wilson procedure—in some work by Friedman and Savage—and suggested that it was different from the author's approach. He continued to outline the points of philosophical difference between his own views and those of the author—particularly on the place of the decision-theoretical approach.

R. C. Curnow commented upon the assumption that "long-run" desirability was useful as a criterion for experimental design—referring also to previous contributions by R. N. Curnow and Yates. M. Stone showed that, when distributions are normal the *D*-, *E*- and *A*-optimalities are special cases of a criterion based upon the Shannon concept of "information" with known variance.

The author replied—amplified at length (in writing)—dealing with the spirit of decision theory in relation to the many aspects and questions raised by the discussion.

(W. R. Buckland)

NAIR, K. R. & KISHEN, K. (Forest Research Institute, India) **9.0** (9.1)

Recent developments in experimental design with special reference to the work in India—*In English*

Bull. Int. Statist. Inst. (1960) **37**, III 161-177 (114 references)

In this paper the authors present a brief review of the developments in experimental design with special reference to the work done in India during the last two decades. The developments include the complete solution of the problem of constructing confounded designs for the general symmetrical factorial experiment, partial solution of the same problem for some important cases of asymmetrical factorial experiments, the introduction of fractional replication of factorial experiments and of incomplete factorials for study of response surfaces, intensive study of the properties and methods of construction and analysis of incomplete block designs of various types such as the balanced incomplete blocks and partially balanced incomplete blocks and the use of these designs for two-way elimination of heterogeneity. The paper also calls attention to new classes of incomplete block designs recently introduced, such as ternary balanced designs, chain block designs, augmented designs, reinforced designs and doubly balanced designs. A development which shows much promise for the future is the use of punched card technique and of electronic computers for routine statistical analysis of replicated experiments.

(K. R. Nair)

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NAIR, K. R. & KISHEN, K. (Report of discussion on a paper by) **9.0** (9.1)

Recent developments in experimental design with special reference to the work in India—*In English*

Bull. Int. Statist. Inst. (1960) **37**, I 118-120

Those contributing to the discussion were: Box, Pearce, and Bose.

Box commented that the author's paper illustrated how the various efficient designs were linked together by very close mathematical relationships. New work with a colleague provided a further illustration: a new type of partial replicate design of the 3^k series had been evolved by choosing a suitable incomplete block design and superimposing upon it a two-level factorial or fractional factorial design. The theory of these designs would be the subject of a forthcoming publication.

Pearce indicated that there was some recent work which would expand section four of the author's paper. Designs proposed by W. Taylor had treatments totally balanced one way and partially the other way. This was helpful, but even then it was probably better to use designs partially balanced both ways.

Bose observed that in very large class of partially balanced designs with two available classes the efficiency of the two types of comparisons were very nearly equal. Hence the difficulty mentioned by Pearce would not arise, or would be negligible, if such designs were used. When partially balanced designs were used for factorial situations it might be better to use those designs in which the efficiencies of the two types of comparison were different and in such a way that the main effects and the lower-order interactions were estimated with greater precision than the higher-order interactions.

(W. R. Buckland)

RAO, C. R. (Indian Statistical Institute, Calcutta) **9.1** (7.1)

Expected values of mean squares in the analysis of incomplete block experiments and some comments based on them—*In English*

Sankhyā (1959) **21**, 327-336 (4 references, 4 tables)

The paper considers the randomisation distribution of the observations in an incomplete block design when the actual layout of the experiment is determined by, first, R_1: assigning the blocks of the plan to the experimental blocks at random and then R_2: allocating the treatments in a block of the plan to the plots of the chosen experimental block, at random, independently for each block.

Under the additive hypothesis, the combined intra- and inter-block analysis is deduced by applying least-squares theory. Incidentally some of the points not fully elaborated in the author's earlier paper [*J. Amer. Statist. Ass.* (1947) **42**, 541] are clarified. Only intra-block estimation is possible when R_1 is not followed and the block assignment is arbitrarily done. In this case the expected values of the " error-mean-square " and the " mean-square for treatments eliminating blocks " are computed under the additive hypothesis. It is shown that a necessary and sufficient condition for their equality is that the sum of the variances of all pair-wise comparisons between varieties occurring in a block of the plan should be the same in all blocks, the variances being computed in the usual way as in the least-squares analysis. This condition is satisfied by the balanced incomplete block designs also by some partially balanced incomplete block designs. The possibility of characterising all designs by the presence or absence of this property is suggested.

The author also obtains the expected values of the treatment and error-mean-squares, under a non-additive model, for a general incomplete block design and also for the special case of a balanced incomplete block design. In the latter case it is shown that the analysis of variance test has a reasonable chance of detecting departures from the null-hypothesis only when the overall differences in yields exceed a certain magnitude depending on the block × treatment interaction. Some standard incomplete block designs are then examined with respect to the terms in which the expectations of mean-squares for varieties and error agree and the extent of disagreement in others. The paper ends with a brief discussion on the effects of random indexing of varieties.

(S. K. Mitra)

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SPRENT, P. (East Malling Research Station, Maidstone, England) **9.0** (6.0)

Analysis of small experiments—*In English*

Nature (1960) **187**, 437-438 (1 reference)

In this note the author briefly considers the effect of departures from assumptions based upon normal distribution theory in connection with significance tests and estimation procedures for small-scale experiments. Three logically distinct but practical situations are considered:

(i) size of experiment limited by number of experimental units available;
(ii) size of experiment limited by other factors: ample experimental units available and those used selected on same basis of uniformity;
(iii) same as with (ii) above but the actual units selected at random.

For the first kind of situation the parent distribution can only be designated from previous knowledge. Although distribution-free randomisation tests by Pitman and others are available [*J. R. Statist. Soc.*, Suppl. (1937) **4**, 119-130 and 225-232] and are appropriate they do not provide tests at a practically useful level. In situations of the second kind, the method of selection for uniformity usually invalidates the normal theory tests. The third kind of situation affords a valid application of normal distribution theory—where the distribution is appropriate: the relevant sampling theory of other distributions may be applied where these can be identified.

In a series of sampling trials, using the ordinary t-test, too high a proportion of significant results was nearly always obtained at conventional significance levels when the basic assumptions did not hold. The author suggests that problems of regression and correlation with only a few pairs of observations would be worth consideration from these points of view.

(W. R. Buckland)

TUKEY, J. W. (Princeton Univ. and Bell Telephone Labs., Murray Hill, N.J.) **9.0** (11.0)
Where do we go from here ?—*In English*
J. Amer. Statist. Ass. (1960) **55**, 80-93 (40 references)

This was the concluding paper at the Symposium of the Design of Industrial Experiments and raises the question, " Along which paths should experimental statistics develop? "

The author attempts to answer this question by suggesting that statisticians should:

(i) recognise that the selection of an " experimental design " is really the selection of a pattern and is only a small part of designing an experiment;
(ii) develop further the possibilities of restricted randomisation in many different kinds of patterns;
(iii) learn to balance conscientiously the importance of bias and variability, especially in regression situations;
(iv) develop new " fractionated " experimental patterns, such as pieces of mixed factorials, which provide desired properties with fewer observations and hence at lower cost by sacrificing some of the efficiency of the estimates;
(v) recognise the effect of distinct aims in diversifying well-chosen methods;
(vi) look to new sources of stimulation such as experiments involving very many factors (random-balance designs), problems of tolerance design and application, and those statistical techniques really appropriate to research; and
(vii) look more and more frequently at broader aspects of experimentation, considering investigations rather than single experiments.

To deal effectively with investigations the author recommends that statisticians should:

(i) consider indications as well as conclusions and keep the two concepts separate;
(ii) work hard on problems of mutual understanding;
(iii) seek new sorts of real problems;
(iv) reshape old tools to new ends;
(v) have a greatly increased concern with problems of choosing the structure within which the analysis is performed, which includes the problems of the

TUKEY, J. W. (Princeton Univ. and Bell Telephone Labs., Murray Hill, N.J.) **9.0** (11.0)
Where do we go from here?—*In English* *continued*
J. Amer. Statist. Ass. (1960) **55**, 80-93 (40 references)

choice of variables, of mode of expression and of representations; and
(vi) learn to use the null hypothesis really appropriate to the situation.

The statistician will also need to step aside and consider other broad fields where such words as experimentation and investigation are not only inappropriate but also dangerous. The development of " evolutionary operation " as an operating tool in production is but one instance of what is possible.

(S. Addelman)

VAART, H. R. van der (Inst. Theoretical Biology, Leiden University) **9.3** (3.9)
On certain types of bias in current methods of response-surface estimation—*In English*
Bull. Int. Statist. Inst. (1960) **37**, III 191-203 (13 references)

Let η represent the yield of some biological or chemical process, depending on k variables $\xi_i (i = 1, \ldots, k)$; a "point in ξ-space" is a set of values of $\xi_1, \ldots, \xi_k$. Assume: (1) η is a polynomial of degree two: $\eta = \eta_0 + \Sigma_i \beta_i \xi_i + \Sigma_i \Sigma_j \phi_{ij} \xi_i \xi_j (\phi_{ij} = \phi_{ji}$; summations are over $1 \leqq i \leqq k$, $1 \leqq j \leqq k$; denote the matrix $\| \phi_{ij} \|$ by $\mathbf{\Phi}$), (2) observations on η are realisations of a random variable y, which has a normal distribution with $\mathscr{E}y = \eta$. Let observations on η be made for N points in ξ-space. Under well-known standard assumptions the least-squares estimators f_{ij} for $\phi_{ij} (1 \leqq i \leqq j \leqq k$; $f_{ji} = f_{ij})$ are multi-normally distributed with $\mathscr{E}f_{ij} = \phi_{ij}$ and covariance matrix depending on the design of these N points. The paper (reporting on work done at the Institute of Statistics, Raleigh, North Carolina) considers (*A*) the question of scaling, (*B*) the distribution of the latent roots of the matrix $\boldsymbol{F} = \| f_{ij} \|$. With reference to (*A*), in defence of Box's rotatable designs which are sometimes criticised for depending on the relative scales for $\xi_1, \xi_2, \ldots, \xi_k$, the paper points out firstly that scale-invariance may be undesirable since the region $\mathscr{R}$ in ξ-space which is of primary interest to the experimenter is bound to influence the nature of the results wanted, secondly, that provided $\mathscr{R}$ may be approximated by a certain type of ellipsoid, rotatable designs permit estimating η with approximately constant variance along the boundary of $\mathscr{R}$. With reference to (*B*), Box and others used to explore response surfaces [see *Biometrics* (1954) **10**, 16-60 and (1955) **11**, 287-323] by writing fitted second degree equations in canonical form [see *Ann. Math. Statist.* (1957) **28**, 239]. Since the latent roots of $\boldsymbol{F}$ constitute the coefficients of this canonical form, their distribution is of interest.

Some results of the paper: firstly under standard assumptions (including normality) and for $k = 2$: the exact distribution of one latent root; and secondly dropping normality and for arbitrary k: certain inequalities for the expectations and medians of the latent roots and certain equalities for their variances and covariances. From these results the paper derives that the above method of exploring response surfaces will unduly often lead to the conclusion that a quadratic

VAART, H. R. van der (Inst. Theoretical Biology, Leiden University) **9.3** (3.9)
On certain types of bias in current methods of response-surface estimation—*In English*
Bull. Int. Statist. Inst. (1960) **37**, III 191-203 (13 references)
continued

surface is of hyperbolic type when in fact it is ellipsoidal. This is interesting since various experimenters have been troubled on finding their analyses indicating a hyperbolic surface when only an ellipsoidal one would seem capable of a plausible interpretation. The result is also relevant to agricultural experiments since the differences for example in weather conditions, between successive years will impair the between-years comparability of data, thus invalidating iteration procedures which in other fields of application serve to correct the bias under discussion.

Two by-products of the investigation: Box & Hunter's statement [*Ann. Math. Statist.* (1957) **28**, 195-241] concerning the variances of the latent roots of $\boldsymbol{F}$ is shown to be incorrect, and with rotatable designs the distribution of the latent roots is shown to be invariant under rotations in ξ-space.

(H. R. van der Vaart)

AKAIKE, H. (Institute of Statistical Mathematics, Tokyo) **10.6** (10.1)

Effect of timing-error on the power spectrum of sampled-data—*In English*

Ann. Inst. Statist. Math., Tokyo (1960) **11**, 145-165 (5 references, 7 figures)

The effect of errors in timing on the shape of the power-spectral function of sampled-data is analysed. It is shown that the sequence of sampled-data forms a stationary stochastic process when the time intervals between the sampling-time points form a stationary stochastic process which is independent of the original process. A general procedure is given for getting the power-spectral distribution function of the sampled-data process. The following two special cases are studied in detail. The first is the case where the errors at pre-assigned sampling-time points, which are the integral multiples of some fixed positive constant, in fact follow one and the same distribution mutually and independently. The other is the case where the sampling points form a renewal process. It is shown for the former case that the timing-error acts as a low-pass filter combined with an inner white noise source. A method of estimation for the level of the white noise is indicated. For the latter case the effect of timing error is represented by using the Poisson's kernel function. It can be seen that the time-sampling is essentially non-linear and the timing-error "whitens" the spectrum. These results are illustrated by using the data obtained by sampling experiments. Two continuous records of the output of an accelerometer mounted on the front axle of an automobile are used to make comparisons of the effect of quantisation error with that of timing error.

[Note: at page 153 on lines 10 and 11 $d\,|\,\phi(\cdot)\,|\,P(\cdot)$ should be read as $|\,\phi(\cdot)\,|\,dP(\cdot)$].

(H. Akaike)

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ANDERSON, T. W. (Columbia Univ. and Cent. Advanc. Studies Behavioural Sci.) **10.7** (10.2)

On asymptotic distributions of estimates of parameters of stochastic difference equations—*In English*

Ann. Math. Statist. (1959) **30**, 676-687 (3 references)

This paper is concerned with the stochastic recurrence relation $x_t = \alpha x_{t-1} + u_t$, $t = 1, 2, \ldots$, where x_0 is a constant, $\mathscr{E}(u_t) = 0$, $\mathscr{E}(u_t^2) = \sigma^2$, $\mathscr{E}(u_t u_s) = 0$, $t \neq s$. An estimator for α is given, and previous studies of the properties of this estimator briefly outlined [see, for example, White, *Ann. Math. Statist.* (1948) **29**, 1188-1197: abstracted in this journal No. 320, **10.7**]. The work presented here is a generalisation of previous work on the distribution of this estimator.

The unstable case $|\,\alpha\,| > 1$ is considered first. A number of theorems are given and proved which lead to several major statements characterising the asymptotic distribution of the α-estimator, call it $\hat{\alpha}$. It is shown that the distribution of the u's will influence the distribution of $\hat{\alpha}$. The standard central limit theorems do not apply here and consequently it is not sufficient to know that the u's are independently distributed to deduce the asymptotic form of the distribution of $\hat{\alpha}$. The author also considers the extension of this problem to the vector case where $\boldsymbol{x}_t$ and $\boldsymbol{u}_t$ are p-component vectors and $\boldsymbol{\alpha} = p \times p$ matrix; analogous results are obtained.

The second part of the paper is concerned with the stable case $|\,\alpha\,| < 1$. Here the asymptotic distribution of α is obtained under rather weak assumptions on the distribution of the u's. The variance of u is assumed finite, but nothing is assumed about the moments of higher order.

(D. H. Shaffer)

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BARTLETT, M. S., GOWER, J. C. & LESLIE, P. H. (Univ. Manchester, Rothamsted Exp. Sta. and Bur. Animal Pop., Oxford) **10.9** (10.2)

A comparison of theoretical and empirical results for some stochastic population models —*In English*

Biometrika (1960) **47**, 1-11 (7 references, 6 tables)

This paper develops in some detail the (approximate) theoretical method of studying the moments of non-linear quantities by means of successive expansion in powers of derivations from the means. The development is carried out in the context of the logistic stochastic model and the other stochastic models of one or more species. The first part of the paper gives the theoretical results: first for single species models—continuous time and discrete time—and then for models with two species. The methods outlined may prove useful not only in the strictly stationary case but also when an extinction probability is small enough for a break-down of stationarity to be ignored. In particular, second-order corrections are given to the mean and third moment of the marginal distribution of population size for two formulations of the logistic model: one in continuous time and one in discrete time. The relevant formulae are summarised at the end of section 3.

In the second part of this paper the theoretical (approximate) results are compared with the exact results in the case of the logistic model in continuous time (Table 1). For the logistic model in discrete time the theoretical (approximate) results are compared with empirical results derived from an artificial series of data (Tables 3, 4, and 5). The agreement was seen to be entirely satisfactory in all cases. The discussion of the two-species model is less detailed but a comparison of first-order theoretical results with the empirical results is shown to be quite good in the case of two competing species fluctuating in the neighbourhood of a stable equilibrium point.

(M. S. Bartlett)

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BURROWS, C. (University College, London) **10.1** (11.1)

Some numerical results for the waiting times in the queue $E_k/M/1$—*In English*

Biometrika (1960) **47**, 202-203 (2 references, 2 tables)

For the queueing situation with general independent input, exponential service-time distribution, and $\delta(\geqq 1)$ servers with the simple first come, first served queue discipline, it has been shown that the equilibrium distribution of queue length is geometric except for modification to the first $(s-1)$ terms [Kendall, D. G., *Ann. Math. Statist.* (1953) **24**, 338-354]. In this note, values for the common ratio of the geometric part of the series are computed when the input is of the Erlangian k phase type for different values of traffic density.

These values are used to compute the ratios of expected time spent waiting for service to the expectation of the time spent in service when there is only one server. These are compared with the ratios obtained when the inter-arrival distribution is exponential while the service-time distribution is not necessarily so.

(C. Burrows)

BOSE, R. C. & KUEBLER, R. R., Jr. (University of North Carolina, Chapel Hill)

A geometry of binary sequences associated with group alphabets in information theory—*In English*

Ann. Math. Statist. (1960) **31**, 113-139 (15 references)

The letters of an (n, k) group alphabet are n-place sequences of 0's and 1's; the alphabet contains 2^k such sequences out of a possible 2^n. The " weight " of a letter is the number of 1's in it. Under the operation of digitwise binary addition these sequences form a sub-group of the group of all possible sequences of n binary digits. When a letter is sent over a channel, noise may change a 0 to a 1 or vice versa. The receiver translates each received sequence into the letter of the alphabet differing from it in the least number of places; this still may not be the letter transmitted, but if it is, the receiver has corrected all the errors introduced by the noise. The aim is to find an (n, k) alphabet in which this procedure can correct as many errors as possible; it is one in which nearest neighbours are as far away from each other as possible, the distance of two letters being defined as the number of places in which they differ. Such " largest-nearest-neighbour-distance " group alphabets are studied here by means of a finite projective geometry, which was originally used by Bose to construct balanced incomplete block designs.

The co-ordinates of a point in the projective space are k-tuples of binary digits 0 and 1; the space contains 2^k-1 such points, omitting (0, 0, ...0). An (n, k) alphabet can be formed by writing down the co-ordinates of each of these points $n_i \geqq 0$ times in adjacent columns, with $\Sigma n_i = n$. The rows of the resulting matrix form k generators of the code sub-group; the rest of the alphabet is obtained by forming all possible linear combinations of these rows or vectors, using binary scalar multiplication and binary vector addition. The problem is then to assign the " n-measure " n_i to the points of the projective space so as to achieve a largest-nearest-neighbour-distance alphabet.

Sets of points whose co-ordinates obey a linear relation form a " flat " in the projective space. A dual projective space is set up, with the points of one space corresponding to flats in the other. It is shown how the assignment of

BOSE, R. C. & KUEBLER, R. R., Jr. (University of North Carolina, Chapel Hill)

A geometry of binary sequences associated with group alphabets in information theory—*In English*

Ann. Math. Statist. (1960) **31**, 113-139 (15 references)

an integral " D-measure " $d_i \geqq 0$ to each point of the dual space is equivalent to assigning an n-measure n_i to the points of the original space. A set of congruences is then derived that the D-measure must satisfy to produce a largest-nearest-neighbour-distance group alphabet, and a new inequality on the number of errors such an alphabet can correct is derived. By number-theoretic arguments using the congruences all largest-nearest-neighbour-distance group alphabets for $k = 2$, 3, and 4 are constructed.

(C. W. Helstrom)

CONSAEL, R. & LAMENS, A. (Ministry of Labour, Brussels) **10.1** (–.–)

Markoffian branching processes in demography—*In French*

Bull. Int. Statist. Inst. (1960) **37**, III 271-289 (17 references)

The authors commence their paper by stating that Kolmogoroff & Dmitrieff studied the most general type of Markoffian multiplicative chains [*C.R. Acad. Sci. U.R.S.S.* (1947) **56**, 5-8] and that they have shown that the machinery of such processes is ruled by two differential equations: these correspond to the backward and forward equations of Kolmogoroff that govern the general Markoffian processes.

This process is resumed, for a single random variable, in the first part of this paper, but this time its structure has been defined by means of an integral equation from which the backward and forward equations may be derived. By particularising this pattern one may find the well-known birth-and-death process, which accounts for the evolution of population strength, and disregards the age distribution of individuals.

Starting from an analogous point of view, the second part of the paper examines an essentially different process. The multiplicative property makes room for an additive structure, enabling an extension of the Pollaczek-Geiringer distribution to the case of a non-homogenous process.

A third type of process, defined as a combination of both the former processes, is surveyed by means of the same technique. As a particular case, it includes the birth-and-death process with effect of immigration.

Finally, in the fourth part of the work it is shown that the method used extends easily to the case of a branching process with two random variables. When working out a birth-and-death process for a population which includes two types of individuals, new difficulties are encountered and are brought to notice.

In an appendix the integral equation, surveyed by Bellman & Harris, is recalled: furthermore, it is shown under which conditions this equation can be used as a means of describing a demographic process.

(R. Consael)

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CANE, Violet R. (Newnham College, Cambridge) **10.1** (–.–)

Behaviour sequences as semi-Markoff chains—*In English*

J. R. Statist. Soc. B (1959) **21**, 36-49 (5 references, 2 tables)

Long records of steady-state behaviour of organisms are usually summarised by giving the relative frequencies p_1, p_2... and mean durations m_1, m_2... of each kind of behaviour. Three models are considered. For the case of only two states these are:—(*E*), the organism alternates between two states, the time spent in each state being independent random variables (with different distributions); (*M*), a Markoff chain model: each instant the organism decides whether to change state or whether to continue in its present state; (*SM*), The semi-Markoff model: the organism stays in one state for a random time interval as in (*E*) and then decides (independently) which state to enter next. The relations between the models are discussed and an interpretation of the (*SM*) model is given. If this model holds, then m_1 is linearly related to p_1/p_2. The generalisation to the case of three or more states are given; if some states are pooled, the (*SM*) model remains of the same form. Another model is constructed for a case in which observations suggest increasing organisation of behaviour.

Three sets of published data on birds, fishes and insects are discussed. A measure of distance between two transition matrices is given. The effect on the theory of taking observations at finitely separated times is discussed. With more data, for example if the variances of bout-lengths were given, it would be possible to distinguish more aspects of the behaviour pattern.

(C. L. Mallows)

CANE, Violet R. (Report of discussion on a paper by)
Behaviour sequences as semi-Markoff chains—*In English*
J. R. Statist. Soc. B (1959) **21**, 49-58

10.1 (–.–)

The contributors to the discussion were Armitage, Audley, Andrew, Weidmann, Cox, Skellam, Webb, Hinde and Stafford Beer: the author replied.

Armitage pointed out that in the (*SM*) model the distribution of bout-length was not completely general, though it may be true that the assumptions (which imply that the bout-length is the sum of r similarly distributed variates—where r is a geometric variate) make a wide variety of forms available. It is to be hoped that more data on this distribution will become available. It is not clear how well established the assumptions are, since the linear relationship used by the author may not be specific to the (*SM*) model.

Audley discussed the utility of probability models for complex behaviour situations; they suggest suitable indices of behaviour, they enable an approach to be made to the evolutionary situations, and they constitute null hypothesis which may suggest better modes of analysis. Models for the non-steady state situations are needed; in some cases time needs to be included as a partly controlling variable. The (*SM*) model may cover the so-called " discontinuity " theories of learning behaviour.

Andrew discussed the concept of a measured " degree of organisation " or " goodness of adaption " of behaviour, this is needed in the design of machines to show adaptive behaviour. Weidmann said that this paper demonstrated the need for co-operation with statisticians. None of the various models is yet established and some of the assumptions made seem questionable. Did the parameters entering into the models correspond to any actual physiological processes underlying behaviour?

Cox related the (*E*) model to renewal theory, and described an application to the study of the precision of systematic sampling. Skellam said that the value of a model lies not only in the closeness of the approximation achieved, but also in the fruitfulness of its concepts. Oversimplification must be avoided.

Webb described an investigation in which his analysis of the data had successfully divided it into two parts, but initial assessment of the relative importance of the parts

CANE, Violet R. (Report of discussion on a paper by)
Behaviour sequences as semi-Markoff chains—*In English*
J. R. Statist. Soc. B (1959) **21**, 49-58

10.1 (–.–)
continued

was the reverse of that of his collaborating psychologist. Hinde distinguished between theories involving purely mathematical relationships between stimuli and responses, and theories which attempt to relate behaviour to associated physiological events. Will the model used by the author assist physiological analysis?

Stafford Beer appealed for a new thought on the appropriate models. Cyberneticians attempt to create a mechanism isomorphous with the animal, and not merely statistically equivalent. No one has discovered a neurological source of random numbers.

(C. L. Mallows)

DARWIN, J. H. (Dept. Sci. Industr. Res., Wellington, New Zealand) **10.3** (10.1)
Note on the comparison of several realisations of a Markoff chain—*In English*
Biometrika (1959) **46**, 412-419 (10 references)

The author considers a simple Markoff chain of s states; that is where the probability of being in state k at time t depends only on the previous state and, given that this is j, the transition probability p_{jk} is independent of t. After N transitions the probability of being in state j tends to P_j as $N \to \infty$. The author remarks that results of Bartlett, for a long realisation of N transitions, of which n_{jk} are from j to k, give the asymptotic equivalence of the distribution of $\{n_{jk}\}$ to a set of s independent multinomial distributions $(n_{j1}, \ldots, n_{js})$, $j = 1, \ldots, s$, generated respectively by the NP_jth power of

$$p_{j1} + \ldots + p_{js}.$$

If there are r long series of transitions it has stated that the likelihood ratio criterion of the equality of the r transition matrices is equivalent to a sum of contingency table χ^2's (the method of proof is named) and has the consequent asymptotic distribution as χ^2 of $(r-1)s(s-1)$ degrees of freedom. The likelihood function maximised is apparently the multinomial one; the $[P_j]$ being treated as if they were known constants and consequently not appearing in the likelihood ratio criterion. Later the probabilities $v_{ij} = p_{ij}/(1-p_{jj})$ of transition, given that there is a change of state, are introduced and the same treatment given to testing

(i) the hypothesis of the equality of the $\{v_{ij}\}$ in the r series and
(ii) that of the equality of the $\{p_{jj}\}$ in the r series.

The two likelihood ratio criterions here are found to sum to the overall one and " by analogy " to have distributions as χ's of $(r-1)s(s-2)$ and $(r-1)s$ degrees of freedom respectively. When the process consists of observations of a continuous process, at time intervals so small that they have negligible chance of including more than one change of state, n_{jj} may be replaced by the time of occupation of the jth state and the test (ii) may be regarded as a test (between r series) of equality of s constants of exponential occupation-time distributions.

The hypothesis that the $\{P_j\}$ are the same in two series is considered, using Bartlett's result [*Proc. Camb. Phil.*

DARWIN, J. H. (Dept. Sci. Industr. Res., Wellington, New Zealand) **10.3** (10.1)
Note on the comparison of several realisations of a Markoff chain—*In English*
Biometrika (1959) **46**, 412-419 (10 references)
continued

Soc. (1951) **47**, 86-95] of the asymptotic normality of the $\{n_{j.}\}$ (where $n_{j.}^{*} = n_{j1} + \ldots + n_{js}$) with mean NP_j and a simplified form of Patankar's asymptotic dispersion matrix [*Biometrika* (1954) **41**, 450-462]. Considering the $\{n_{j.}\}$ for the two series as multinomial variables the likelihood ratio criterion for common means, assuming the dispersion matrix known, is constructed and the efficient estimate of the dispersion matrix from the pooled series substituted for it to give a computable likelihood ratio criterion. This has asymptotic distribution as χ^2 of $s-1$ degrees of freedom. It is considered in more detail for $s = 2$ and then straightforwardly generalised for any s to the case of r series—having asymptotic distribution as χ^2 with $(r-1)(s-1)$ degrees of freedom.

Finally, some educational data ($r = 2$, $s = 6$ and N about 1,000) is analysed in respect of equality between the $\{p_{ij}\}$, the $\{v_{ij}\}$ and the $\{p_{jj}\}$.

(D. E. Barton)

DURBIN, J. (London School of Economics) **10.2** (10.7)

Efficient estimation of parameters in moving average models—*In English*

Biometrika (1959) **46**, 306-316 (9 references, 1 table)

A time series of n observations $x_1, \ldots, x_n$, equally spaced in time, is supposed to have moving average structure. For simplicity of exposition the first order moving average is treated initially, where $x_t = \varepsilon_t + \beta\varepsilon_{t-1}$ and the ε's are independent $N(o, \sigma)$ variates. It is noted that even Whittle's (1951) first order approximation to the maximum likelihood equation (for $\hat{\beta}$) involves the extraction of the root of a high order polynomial (whose coefficients are the sample serial correlations). The author first recalls the result that for any integer k, x_t may be expressed as $\varepsilon_t + l_k +$ residual where l_k is a linear form in the k previous x's (whose coefficients are powers of β) and the residual is of order the kth power of β. Hence, choosing k large (but small relative to n) the residual may be neglected and the scheme thus replaced approximately by an autoregressive one. Next it is recalled that Mann & Wald [*Econometrica* (1943) **11**, 173-220] showed that the estimators $a_1, \ldots, a_k$ of the coefficients in the autoregressive scheme $x_t = L_k + \varepsilon_t$ (where $L_k = \alpha_1 x_{t-1} + \ldots + \alpha_k x_{t-k}$), obtained by minimising the sum of squares of $x_t - L_k$, are asymptotically normal with respective means $\alpha_1, \ldots, \alpha_k$ and specified dispersion matrix. The author considers the normal asymptotic likelihood of the a's and, maximising this with respect to β, he obtains the estimator b of β which is minus the first-order sample serial correlation coefficient of the sequence $a_1, \ldots, a_k$, he again neglects terms of small order in n. He remarks that this converges in probability to a quantity which differs from β by terms of order the kth power of β and its variance differs similarly from $(1-\beta^2)/n$, this Whittle [*Ark. Mat.* (1953) **2**, 423-434] showed to be a lower bound for consistent estimators of β.

Proceeding now to hth order moving averages of coefficients $\beta_1, \ldots, \beta_h$, he uses the same device to obtain estimators $b_1, \ldots, b_n$ which are the solutions of k simultaneous linear equations whose coefficients are the sample serial correlations of $a_1, \ldots, a_k$. The asymptotic dispersion matrix of these is obtained in terms of $\beta_1, \ldots, \beta_n$, only approximately and for k large. Next, the hypothesis that $\beta_1, \ldots, \beta_n$ have specified values is considered and the usual χ^2 of h degrees-of-freedom test,

DURBIN, J. (London School of Economics) **10.2** (10.7) *continued*

Efficient estimation of parameters in moving average models—*In English*

Biometrika (1959) **46**, 306-316 (9 references, 1 table)

that is a multiple of the exponent of the multi-normal approximate distribution of $b_1, \ldots, b_n$, is proposed. A test of the "fit" of the hth order moving average is next considered namely, the minimal sum of squares of Mann & Wald which is, under the null hypothesis, distributed approximately as a χ^2 of $k-h$ degrees of freedom. This test function is shown to have a simply computed form.

Finally a sampling experiment is reported, in regard of the first order moving average with $\beta = \frac{1}{2}$, based on 20 runs of $n = 100$. Good agreement for the sampling variance of b was obtained, but b itself was found to average at 0·45 which, being well below twice its standard error, was adjudged as indicating a bias of the order of 10 per cent.

(D. E. Barton)

DURBIN, J. (London School of Economics) **10.2** (10.7)

Estimation of parameters in time series regression models—*In English*

J. R. Statist. Soc. B (1960) **22**, 139-153 (9 references)

A time series $y_1, y_2, \ldots$ has, for each y_t, a set of q concomitant variables $x_t \equiv (x_{1t}, \ldots, x_{qt})$. A mixed regression and auto-regression model is set up: $y_t + \alpha_1 y_{t-1} + \ldots + \alpha_p y_{t-p} = \beta_1 x_{1t} + \ldots + \beta_q x_{qt} + \varepsilon_t$, where the $\{\varepsilon_t\}$ are unobservable, independent random variables. Least-squares estimation of the (unknown) α's and β's is discussed with reference to the work of Anderson & Rubin [*Ann. Math. Statist.* (1950) **21**, 570-582] which is judged, on several counts, to be inapplicable to the present problem.

Extensions of the concepts of "unbiased" and "minimum variance" estimators are introduced, first, for clarity, in the one parameter case. Namely, if α is a parameter and T_1, T_2 are two functions of the sample such that $\mathscr{E}(\alpha T_1 + T_2) = 0$ then the equation $aT_1 + T_2 = 0$, yielding an estimator a of α, is called an unbiased linear estimating equation. If also $\mathscr{E}(T_1) = 1$ and Var $(\alpha T_1 + T_2)$ is least for all such equations then it is called a *best* unbiased linear estimating equation. Next a lower bound for Var $(\alpha T_1 + T_2)$ is obtained which is analogous in proof and statement to the Cramér-Rao theorem. It is noted that if T_1 converges stochastically on 1, $\alpha - a$ behaves asymptotically as $\alpha T_1 + T_2$ so that in such cases the best unbiased linear estimating equation criterion may be fairly considered as a generalisation of minimum variance relative to a class of estimators.

The two definitions are then extended to the case of several (k) parameters and a generalisation of Rao's matrix "lower bound" theorem obtained. When the likelihood is quadratic in the k parameters it is shown that the likelihood equations are unbiased linear estimating equations. A possible extension to non-linear equations (due to Barnard) with the corresponding lower bound theorem, is sketched. The general results are then used to qualify the least-squares equations: these are maximum likelihood and hence best unbiased linear estimating equations when the ε's are $N(0, \sigma)$ variables in the time series model but the author remarks that it may be "difficult to verify" the stochastic convergence of the multi-parameter analogue of T_1 depending as it may on the behaviour of $x_1, x_2, \ldots$, and on the magnitudes of $\alpha_1, \ldots, \alpha_p$), means that the foregoing results

DURBIN, J. (London School of Economics) **10.2** (10.7)

Estimation of parameters in time series regression models—*In English*

J. R. Statist. Soc. B (1960) **22**, 139-153 (9 references)

continued

may have to be regarded as "approximate". The minimum sum-of-squares, divided by $n-q-p$ is shown to be asymptotically unbiased for Var ε, whether ε is normally distributed or not. Next, restrictions on the magnitudes of $[x_t]$, $[y_t]$, $[\alpha_1]$ and $[\text{Var } \varepsilon_t]$ are made and following Mann & Wald's theory of the auto-regression model the standardised estimators of α's and β's are shown to be asymptotically normal.

Finally the treatment is sketched for a somewhat different model where $y_t = \beta_1 x_{1t} + \ldots + \beta_q x_{qt} + u_t$; $u_t + \alpha_1 u_{t-1} + \ldots + \alpha_p u_{t-p} = \varepsilon_t$, shown equivalent to

$$(y_t + \alpha_1 y_{t-1} + \ldots + \alpha_p y_{t-p}) -$$

$$- (\beta_1 x_{1t} + \ldots + \beta_q x_{qt} + \alpha_1 \beta_1 x_{1t-1} + \ldots + \alpha_p \beta_q x_{qt-p}) = \varepsilon_t,$$

which has non-linear least-square estimators. The method is two-stage: first the coefficients of $x_{1t-1}, \ldots, x_{qt-p}$ are treated as independent parameters and the least-square estimators obtained by minimising the sum of squares of the left hand side of the latter equation are solved for estimators $\{a_i\}$ of $\alpha_1, \ldots, \alpha_p$. Second, the sums $v_t = y_t + a_1 y_{t-1} + \ldots + a_p y_{t-p}$, $w_{it} = x_{it} + a_1 x_{it-1} + \ldots + a_p x_{it-p}$ are formed and the sum of squares of $v_t - (\beta_1 w_{1t} + \ldots + \beta_q w_{qt})$ is minimised for estimators of the β's. An outline proof is given that these estimators have the same asymptotic dispersion matrix as those from the non-linear least-square estimations.

(D. E. Barton)

FINCH, P. D. (London School of Economics) **10.4** (–.–)

Cyclic queues with feedback—*In English*

J. R. Statist. Soc. B (1959) **21**, 153-157 (3 references)

A set of m servers in series is considered, that is " phase type queueing " and two cases are treated: firstly " terminal feedback " where a customer, on completing his service at the last server, has a probability p_j of being returned to the jth server (with $p_1 + \ldots + p_m < 1$) and secondly " single service feedback " where a customer on completing his jth service has a probability p_j, $p_j < 1$, of being returned to the same server. In cyclic queues, customers departing from the mth server are fed back to the beginning so that the processes discussed may be looked at as modified forms of such. Customers arrive randomly at the service system (with rate λ) and service is random (with respective " rates " μ_j). The number of customers present is never greater than N: customers baulk if they find, on arrival, N already in the system. The joint probability density function of the numbers n_j, $j = 1, \ldots, m$ in the queues, customers being served included, is considered; only the equilibrium distribution which is known to exist since the process is an irreducible aperiodic Markoff chain is treated.

The usual equilibrium difference equations are set up for $p(n_1, \ldots, n_m)$ in both cases and the author remarks that these are satisfied by probabilities of the form constant $\times x_1^{n_1} \ldots x_m^{n_m}$ where x_j is a simple multiple and depends only on $p_1, \ldots, p_m$ of λ/μ_j; and that the uniqueness of this solution follows along the lines of Jackson's results [*J. R. Statist. Soc.* B (1956) **19**, 129-132]. The author considers the fact that Jackson's solution of the phase type queueing problem is obtained from those given here for terminal feedback when the limit as $N \to \infty$, $p_j \to 0$.

(D. E. Barton)

FINCH, P. D. (London School of Economics) **10.1** (10.2)

Deterministic customer impatience in the queueing system $GI/M/1$—*In English*

Biometrika (1960) **47**, 45-52 (5 references)

The system considered is that where customers, whose inter-arrival time distribution is arbitrary, arrive at a single server and are served on a first come, first served basis. It is supposed that the service time distribution is of the Poisson type. If a customer has not commenced service before a fixed time W spent in the queue he departs never to return.

By considering the waiting time of the nth customer and relating it to that of the $(n+1)$st a limiting distribution of the waiting time is found and the case where the arrival distribution is of the Erlangian k phase type is given as an example.

The relation to the case when $w = \infty$ is considered and the Pollaczek-Khintchine formula for the Laplace transform of the waiting time distribution is deduced.

A section is devoted to the limiting distribution of queue size at the instants a service commences. Expressions are given for the probabilities that there are n people in the queue and particular expressions for the case where the input process is Poisson.

In the final section the method for modifying previous results if there are more than one server is discussed.

(C. Burrows)

HAIGHT, F. A. & BREUER, M. A. (University of California, Los Angeles) **10.4** (2.1)

The Borel-Tanner distribution—*In English*

Biometrika (1960) **47**, 143-150 (4 references, 1 table)

The Borel-Tanner distribution [Borel: *C. R. Acad. Sci., Paris* (1942) 452-456 and Tanner: *Biometrika* (1953) **40**, 58-69] occurs in the theory of queues as the probability distribution function, $p(x, r, \alpha)$ on the number of people served before a queue first vanishes. The assumptions are an initial queue length of r members, Poisson arrivals with intensity α and constant unit service time.

The first two moments of the distribution are given explicitly and the moment generating function derived. It may be shown that the maximum likelihood estimate of α is equivalent to the method of moments. When $\alpha > 1$, the authors show that there is a finite probability that the queue will never vanish and obtain an expression for $p(x, r, \alpha)$.

The cumulative distribution function, $P(x, r, \alpha)$, is tabled for $r = 1$, $\alpha = 0{\cdot}63(0{\cdot}01)0{\cdot}99$. Certain other tables described in the text are not published but are available from the authors.

(J. G. Saw)

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HANNAN, E. J. (Canberra University College, Australia) **10.2** (10.6)

The estimation of seasonal variation—*In English*

Aust. J. Statist. (1960) **2**, 1-15, (6 references, 4 tables, 1 figure)

The problem considered in this paper is that of estimating seasonal variation in a time series in the circumstances found typically in economic data, where low frequency components of the "noise" superimposed on the seasonal component, are very important.

Economists, it is said, commonly employ a technique based on the use first of a filter which does not modify the seasonal component, followed by a least-squares estimation of the seasonal component. After the estimated seasonal component has been removed the series may be refiltered (using a more suitable filter) and the seasonal component re-estimated. The author points out that such iterative techniques are unnecessary since *any* filter may be used in the first place, its effect upon the seasonal component being known. For monthly data the filter will affect the harmonic component of the seasonal component having frequency $\lambda_j = 2\pi_j/12$, by multiplying its coefficient by $g(\lambda_j)$, where $g(\lambda)$ depends upon the filter used. The values of $g(\lambda_j)$ are tabulated for filters based on Spencers 15 point and 21 point formulas and second and fourth differences.

The estimation of the variance of the estimates of the coefficient of the harmonic components of the seasonal requires the estimation of the spectral density of the noise, after filtering, at the frequencies λ_j, and this is discussed in relation to the type of filter used. These techniques are applied to a series relating to Advances and Discounts for the major Australian private banks over the period 1954 to 1959.

In a final section the discussion is extended to the case where the seasonal variation contains an additional component containing terms $t \cos \lambda_j t$ and $t \sin \lambda_j t$, where t is the time from the beginning of the observed series.

(E. J. Hannan)

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HEATHCOTE, C. R. & MOYAL, J. E. (Australian Nat. Univ., Canberra) **10.4** (10.1

The random walk (in continuous time) and its application to the theory of queues—*In English*

Biometrika (1959) **46**, 400-411 (20 references)

The purpose of this paper is to present a unified development of one-dimensional random walk theory with 0, 1, or 2 barriers, reflecting or absorbing, together with its applications to queueing theory when service and arrivals are both random and, by the usual limiting process, to diffusion theory.

A particle has uniform chances, per unit time, of moving to the right or the left by amounts λ, μ respectively. The usual " backward " equations are set up for the probability, $\Pr_{kn}(t)$ of having moved to a position n steps to the right of the origin at time t, having started k to the right, which are recurrent in k and differential in t. These are then generated by z^n and Laplace transformed by e^{-st} which results in a simple linear recurrence in k for the transform $g_k(z, s)$. The essential uniqueness of the solution of this is deduced from results published elsewhere by Moyal [*Acta Math., Stockh.* (1957) **98**, 221-264].

Two limiting processes are then described; as $t\to\infty$ with a corresponding limit for the Laplace transform, as $s\to+0$ and " passage to a diffusion process ". Next the recurrence is solved generally for $g_k(z, s)$ with the corresponding diffusion result and the particular forms of these in the no-barrier case are obtained. Then follow firstly a form of the solution for two absorbing barriers at 0 and N; secondly the odds of absorption at either barrier; lastly the mean time before absorption. Letting $N\to\infty$ gives the solution for one absorbing barrier: the first passage-time distribution when $k = 0$; its diffusion form and the diffusion distribution of position at time t, having started at x, conditional on non-absorption. Reflecting barriers at 0 and N are then treated; expressions for $\Pr_{k,a}(t)$ and mean n are obtained. Since this is equivalent to a simple queue whose size is limited to N, the mean queue length as $t\to\infty$ and busy-period distribution follow. When $N\to\infty$ the queue length distribution and its mean are obtained explicitly.

Lastly, queues with N servers are considered. The equation for $g_k(z, t)$ takes different forms $k\leqq N$, $k\geqq N$. The solution is obtained in terms of confluent hypergeometric functions and the known limiting distribution as $t\to\infty$ follows. The moments of the busy period also are found.

2/197 (D. E. Barton)

KITAGAWA, T. (Kyusyu University, Japan) **10.1** (8.9)

Successive process of statistical controls. III—*In English*

Mem. Fac. Sci., A, **Kyusyu** (1960) **14**, 1-33 (9 references, 1 figure)

This is the third paper of a series with the same title by the present author [*Mem. Fac. Sci.*, A, *Kyusyu* (1959) **13**, 1-16; abstracted in this journal No. 175, **1.7**]. In this paper some elementary manipulations of the transition probability matrices of finite Markoff chains are exemplified for some special cases to get compact representations of their characteristic equations. The well-known theorem is cited, which states that, by solving the characteristic equations, one can get the representations of the powers of the transition probability matrices using their characteristic values. The author briefly states that following this general approach, one can analyse statistical properties of the control chart method using runs from the point of view of finite Markoff chain. Explicit numerical results are not given in this paper but deferred to a later occasion.

(H. Akaike)

JOWETT, G. A. & WRIGHT, Wendy M. (University of Sheffield) **10.3** (10.7)
Jump analysis—*In English*
Biometrika (1959) **46**, 386-399 (5 references, 6 tables, 1 figure)

A time series $x(t)$ $(t = 1, \ldots, kn)$ is supposed to have the structure $x(t) = \mu(t)+\xi(t)$, where $\xi(t)$ is a stationary time series and $\mu(t)$ is constantly μ_s in the sth " section " of time: $(s-1)n<t\leqq sn$, $s = 1, \ldots, k$. The problem is to test the hypothesis that μ_s does not change from section to section, the auto-correlated nature of the series invalidating an analysis of variance between and within sections. An integer α is chosen, $\alpha<\frac{1}{2}n$, and the " between section jump statistic " B_α is then computed, being, essentially, the sum over the $k-1$ pairs of successive sections of the squared difference between the means of the last α observations in the first and of the first α in the second section of a pair. The null hypothesis mean and variance of this are found, as functions of the unknown autocorrelation function and also, to the first order in $1/k$, those of log B_α: the mean being log $\mathscr{E}(B_\alpha)$ to this order. In particular $\mathscr{E}(B_\alpha)$ is shown to depend on the autocorrelation function for time lag in an interval of $\pm 2\alpha$ whilst the same is true of the major parts of the variance of B_α and the variance of log B_α: that of the latter being, in fact, $2/(k-1)$ and independent of the autocorrelation function.

There then follows a general discussion of various assumptions which ensure that the minor part of these variances is negligible and of methods of estimating it when it is not. An estimator W_α of $\mathscr{E}(B_\alpha)$ is proposed, quadratic in within-section deviations from section means and shown to be unbiased. The statistic $J_\alpha = \log (B_\alpha/W_\alpha)$ is defined and, to order l/k, its mean is shown to be zero and an expression found for its variance. The leading coefficient in this variance is computed for $n = 12$, $\alpha = 1(1)6$, $p = 0{\cdot}0\ (0{\cdot}1)\ 0{\cdot}9, 0{\cdot}95$ in the case of the exponential autocorrelation function, $\rho^{|\tau|}$ where τ is the lag and p a constant, and from this it is concluded that $(k-1)$ Var J_α may be taken as 2·2 for $\alpha\leqq 6$: as ρ increases the optimal value of α increases. A small sampling experiment is reported showing apparent normality for J_α at $n = 6, k = 10$ and $\alpha = 1, 2, 3$ as $\rho = 0{\cdot}8, 0{\cdot}36, 0{\cdot}2$ respectively.

Comparison is made with the analysis of variance

JOWETT, G. A. & WRIGHT, Wendy M. (University of Sheffield) **10.3** (10.7)
Jump analysis—*In English*
Biometrika (1959) **46**, 386-399 (5 references, 6 tables, 1 figure)
continued

when $\mu_1, \ldots, \mu_k$ are normal variables with variance σ_μ^2. that is a random model with null hypothesis $\sigma_r = 0$ and for large k assuming the autocorrelation function is fairly completely known, this implies *inter alia*, that α has been chosen optimally. This is done in terms of the relative efficiencies of the estimators of σ_μ^2 involved in the analysis of variance and, as it is shown, in $B_\alpha - \mathscr{E}(W_\alpha)$. It appears that B_α has a relative efficiency of about 2/3 except when ρ is large (of order 0·95) and σ_μ^2 is of the same order as the variance within sections.

The paper concludes with a worked example and includes a *post facto* discussion of the validity of the analysis based on a scrutiny of the variogram.

Throughout the paper there is discussion in rather general terms of the robustness of the technique in relation to departures from normality, the exponential autocorrelation function, stationarity, the limit as $k\to\infty$.

(D. E. Barton)

KENDALL, D. G. (Magdalen College, Oxford)
Birth-and-death processes, and the theory of carcinogenesis—*In English*
Biometrika (1960) **47**, 13-21 (6 references)

This paper aims at giving a quantative model for a current theory of carcinogenesis. It is supposed that there is an infinite population of normal cells subject to carcinogenic action, for example, irradiation, usually of two kinds; a " background " which is always present and an experimental enhancement of this of known intensity acting over a known period. It is assumed that the effects of these agents transforms some of the normal cells into first-order mutants (" grey " cells). In a given interval of time the number of grey cells produced is taken as a Poisson variable. Each grey cell is supposed to generate a clone, the individuals in which multiply independently as in the simple birth-and-death process with death rate greater than birth rate. It is supposed that owing to the carcinogenic agents grey cells may be transformed into second-order mutants (" black " cells) and such mutations occur independently of the birth-and-death process controlling the growth of grey cell clones. Black cells also develop clones (malignant growth) according to a birth-and-death process but this time the birth rate is greater than the death rate. The experiment will consist of counting black and grey clones after the exposure period, but since all clones are not certain to be counted the probability that a clone of size n is counted is introduced.

The history of a single grey clone descended from a single grey cell formed at the epoch $t = 0$ is then considered. A double generating function, 0, is introduced for the joint distribution of the number of grey cells and the number of detected black clones. An integral equation for 0 is obtained which is reduced to a differential equation of the Ricatti type but without constant coefficients.

Next the problem of the formation of grey clones and the black clones derived from them is considered. The generating function for the joint distribution of the total numbers of detectable grey and black clones is found and from this the means, variances and covariances are given. Some asymptotic formulae are also given.

Finally, some light is thrown on the problem of the so-called " threshold " effect. It is assumed that the rate of the second-order mutations is independent of the

KENDALL, D. G. (Magdalen College, Oxford)
Birth-and-death processes, and the theory of carcinogenesis—*In English*
Biometrika (1960) **47**, 13-21 (6 references)

intensity of the carcinogenic agents then it is shown that the incidence of tumours is proportional to this intensity. However, if the rate of second-order mutations is assumed to be proportional to the intensity of the carcinogenic agents, the rates imply a difference in the character of the response at low and high intensities and it can be said that there is a sort of threshold.

(C. Burrows)

In chapter one of this paper the properties concerning the convergence and stability of the product and division of the independent and correlated random variables are studied by means of the characteristic function and the integral transforms of Mellin. Some new distributions are found.

The second chapter deals with a new stochastic process of addition of random variables in a random manner. It is generalised to the case of correlated multivariate distributions, both on the result of the addition and in the number of summands.

In chapter three the former process is generalised to the product of random variables and a new stochastic process, which is called " integration of random variables between random bounds ", is defined. In the remainder of this paper the properties of this process are studied as well as the properties resulting from its inversion.

In chapter four, by means of the integral transforms of Laplace and Fourier, the properties of the isotropic random vectorial distributions in Euclidean spaces of dimension one are studied: as well as the properties of their projections on sub-spaces of dimension $m(m<n)$ and, in particular, on an axis. The author solves the equations that relate the frequency functions and the characteristic functions. The stable laws of probability, which correspond for these distributions to the Gauss-Laplace and Cauchy laws for scalar variables, are investigated: Solutions are given for some problems on Brownian motion.

Chapter five deals with the different applications of the theories developed in the previous chapters; namely, runs on random size samples, random uniform repartition on the surface and inside hyperspheres, inverse probabilities in stochastic processes, random straight motion, Brownian motion of translation and rotation, contagion with a random initial number of patients, superposition of random variables, etc.

In chapter six the properties of periodic univariate and bivariate random variables are studied. Some applications of the latter are made to the research of random motion on cylinders. In chapter seven the possibility of the existence of relations of uncertainty

which generalised those of Heisenberg and quantum mechanics is shown. A stochastic process with applications to biology and radioactivity is solved.

In the final chapter a scheme of chain probabilities, which is the analytical planning of an event arising from population genetics, is solved.

(S. Rios)

MARAVALL, D. (Inst. Nacional Investigaciones Agron., Madrid) **10.1** (10.9)
Some new stochastic processes—*In Spanish*
Rev. Acad. Madrid (1959) **53**, 130 pages (23 references)

Three new two-dimensional stochastic processes are studied with applications to biology and physics. If a Poisson process involving time as a random variable is inverted then several stochastic processes can be developed: these are studied in this paper—the asymptotic distribution is obtained of the number of draws resulting from n trials of a " head-and-tail " game.

Other problems investigated are:

(i) the deviation and classification of the central limit theorem relating to random numbers and the addition random variables;
(ii) identical replacement—with new and reconditioned units;
(iii) joint distribution and the correlation coefficient between the queue-length and waiting time;
(iv) Brownian motion in n-dimensional Euclidean spaces;
(v) simultaneous and Brownian motion radioactive disintegration: some properties of gamma and Bessel functions are indirectly obtained.

The frequency function is given for the module of isotopic random vectors in an n-dimensional space on which the projections on an axis forms a stable variable and the properties of the additions of isotopic random vectors with constant length and in the same plane are investigated.

The properties of a stochastic filtration process are applied to problems of physics and biology and the partial differential equation of the frequency function of monotonic diffusion processes is obtained: Pareto's law is forecast theoretically. The theory of the stochastic pendulum is developed as a solution for differential equations of first and second order with random coefficients. This leads to a study of the properties of stochastic pendulum motions with impulses.

The frequency function and the characteristic function of the product of two correlated normal variables is obtained in generalised terms. The properties of a probability distribution which is not normalised are investigated. In this context the term " normalised "

MARAVALL, D. (Inst. Nacional Investigaciones Agron., Madrid) **10.1** (10.9)
Some new stochastic processes—*In Spanish*
Rev. Acad. Madrid (1959) **53**, 130 pages (23 references)
continued

indicates that the characteristic function is equal to unity at the zero point. Where considerations of convergence in probability are concerned, the different behaviour of such distributions in relation to ordinary distributions is investigated. Some examples are given and Buffon's needle problem solved for n-dimensional Euclidean space.

(S. Rios)

MARAVALL, D. (Inst. Nacional Investigaciones Agron., Madrid) **10.1** (2.1)
New theorems on Brownian motion and stochastic oscillation—*In Spanish*
Rev. Mat. Hisp.-Amer. (1959) **19**, 51-67 (6 references)

This paper is a study of the properties of pendulum stochastic motion and random motion where the impulses are defined by monotonic stochastic processes. The characteristic function of the joint distribution of the position and speed of the pendulum is obtained. The convergence in probability of these stochastic motions is investigated and some new theorems obtained on the Brownian motion of translation and rotation.

(S. Rios)

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SCHNEIDER, B. (Inst. Mech. Verfahrenstechnik, Techn. Hochsch., Karlsruhe) **10.0** (10.1)
Contribution to the theory of branching processes—*In German*
Metrika (1960) **3**, 128-142 (20 references)

In this paper the author considers frequency densities of the parameter x by introducing the derivative $z(x, t)$ of $Z(x, t)$ with respect to x. That is say, the stochastic distribution $z(x, t)$, which in itself is a second-order stochastic vector process is used as an alternative to the treatment of point processes $Z(x, t)$ by higher-order product-densities.

With the aid of additive random set functions, necessary and sufficient conditions are derived for the differentiability of a given random set function $Z(B)$, that is for the construction of its stochastic distribution $z(x)$. If the covariances of $Z(B)$ can be expressed as a Lebesgue-Stieltjes integral $\mathscr{E}[Z_i(B_1)Z_j(B_2)] = \iint d^2 r_{ij}(x, x')$ over B_1, B_2, where the variation of the covariance functions r_{ij} is restricted, then the relation $Z(I) = V_I z(x)$ holds for all bounded intervals I, where V_I is the operator of variations within I, and $Z(B)$ is mean square differentiable if r_{ij} is generalised differentiable.

(R. Wette)

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PRABHU, N. U. (Kartanak University, Dharwar, India)

10.4 (0.4)

Application of storage theory to queues with Poisson arrivals—*In English*

Ann. Math. Statist. (1960) **31**, 475-482 (20 references)

The following is the author's summary in its entirety " This paper is concerned with the waiting time process, $W(t)$, for the queueing system in which firstly there is only one counter, secondly the customers arrive at random and are served in the order of arrival, and lastly the service time distribution has a general form. It is observed that the Pollaczek-Khintchine formula for the transform of the limiting distribution of $W(t)$ is similar to the one occurring in the theory of continuous time-storage processes, and it is inverted by the method used in that theory. Further, $W(t)$ is shown to be a special case of the storage process, and known methods and results of storage theory are used to obtain the transition distribution function of $W(t)$."

It is pointed out that in the discrete-time model for the dam, the waiting time, $W(t)$, of a customer in the queueing process described above is a temporally homogeneous Markoff process; as is the storage, $Z(t)$. The Laplace transform of the distribution of $W(t)$, when the queue is in " statistical equilibrium " is given by the Pollaczek-Khintchine formula:

$$\text{(i)} \qquad \phi^*(\theta) = \frac{(1-\rho)\theta}{\theta-\lambda+\lambda\psi(\theta)}, \quad R(\theta)\geqq 0$$

where $\psi(\theta)$ is the Laplace transform of the service time distribution $dB(t)$, and $\rho = -\lambda\psi'(0)$ is the relative traffic intensity measured in erlangs, and it is assumed that $\rho<1$.

If the transition distribution function of $W(t)$ is denoted by $F(z, t)$, and $F^*(z) = \lim_{t\to\infty} F(z, t)$ then $\phi^*(\theta)$ is the Laplace transform of $F^*(z)$. Equation (i) is then inverted to give

$$\text{(ii)} \qquad F^*(z) = 1-(1-\rho)\int_0^\infty dK(z+t, t)$$

where

$$\text{(iii)} \quad dK(x, t) = \sum_{n=0}^{\infty} e^{-\lambda t}(\lambda t)^n/n!\, dB_n(x), \quad 0\leqq x<\infty$$

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PRABHU, N. U. (Kartanak University, Dharwar, India)

10.4 (0.4) *continued*

Application of storage theory to queues with Poisson arrivals—*In English*

Ann. Math. Statist. (1960) **31**, 475-482 (20 references)

$B_n(x)$ is the n-fold convolution of $B(x)$ with itself, and $B_0(x) = 0$ if $x<0$ and $= 1$ if $x\geqq 0$. The derivation of these results directly from (i) allows the author to establish a correspondence between the stated queueing problem and a storage process where the input has the distribution (iii). The forward Kolmogoroff equation for $W(t)$ is then treated using methods of the theory of storage processes, giving

$$\text{(iv)} \quad F(z, t) = K(t+z-z_0, t) - \int_0^t F(0, t-\tau)dK(\tau+z, \tau)$$

where $z_0 = W(0)$.

(R. M. Durstine)

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SEYFFERT, W. (Botanic Institute, University of Cologne) **10.9** (10.1)

Theoretical studies on the composition of tetrasomic populations.
I. Random mating—*In German*

Biom. Zeit. (1960) **2**, 1-44 (48 references, 7 tables, 12 figures)

This paper deals with the mathematical genetics of infinite panmictic tetrasomic populations (non-overlapping generations) and is particularly concerned with the rate of approach to population equilibria and their composition. Emphasis is laid on the difference between tetrasomic and disomic (diploid) inheritance throughout.

In two introductory paragraphs the basic mathematical genetics of tetrasomic meiosis is laid out. The probability of double reduction is confined to the limits $0 \leqq \alpha \leqq 1/7$, using the Fisher-Mather definition of the probability of genetic non-disjunction as referring to tetradic segregation by chromatids. A maximum-likelihood estimator for the parameter α of double reduction is derived. The equations governing the gametic output of tetrasomics are generalised for multiple alleles and segregation by chromatids.

In the main part of the paper the following results are derived: dependence of approach to gametic genotype equilibrium on the intrinsic rate of approach to equilibrium, which equals $(1-\alpha)/3$ per generation, and the distance from equilibrium of the actual population, whence follows that the overall rate of approach to equilibrium is larger for segregation by chromatids. Instant gametic equilibrium will be attained under segregation by chromosomes in simplex-triplex-type crosses. The composition of the tetrasomic gametic equilibrium population is identical with the disomic zygotic equilibrium when segregation is by chromosomes and in the case of segregation by chromatids it is identical with a disomic zygotic equilibrium when its population coefficient of inbreeding is $F = 1/5$. The proportion of heterozygotes in tetrasomic gametic equilibrium populations is higher than in disomic zygotic equilibria and may reach 98 per cent. for four alleles and segregation by chromosomes at one locus, it is reduced due to double reduction when segregation is by chromatids. The intensity of selection in tetrasomics is shown to depend not only on the gene frequencies but also on the distance of the initial population from equilibrium; this distance is changing under selection.

2/211 (R. Wette)

SEYFFERT, W. (Botanic Institute, University of Cologne) **10.9** (10.1)

Theoretical studies on the composition of tetrasomic populations.
II. Selfing—*In German*

Zeit. Vererbungsl. (1959) **90**, 356-374 (42 references, 8 tables, 4 figures)

This paper contains the author's study of tetrasomic populations [see *Biom. Zeit.* (1960) **2**, 1-44; abstracted in this present journal No. 2/211 **10.9**]. In this second paper the investigation is of infinite self-fertilising tetrasomic populations with non-overlapping generations by means of Fisher's generation matrix method. Without selection the dominant latent root of the tetrasomic generation matrix is equal to $\lambda_D = (5-2\alpha)/6$, where $0 \leqq \alpha \leqq 1/7$ is the probability of double reduction, whence follows that the rate of approach to homozygosity is decidedly lower than in selfing disomics and lowest with segregation by chromosomes. Under selection against homoallelic zygotic or gonic genotypes the maintenance of heterozygosity is even more pronounced and is highest when segregation is by chromosomes, due to interaction between selection and double reduction.

(R. Wette)

STEINHAUS, H. & URBANIK, K. (Math. Inst., Polish Academy of Sciences, Warsaw) **10.1** (2.5)
Poisson sequences—*In German*
Math. Zeit. (1960) **72**, 127-145 (9 references)

The authors study the concept of a Poisson process with the concept of a Poisson sequence (of real numbers). This study is motivated by the desire to have models of the Poisson sequences as appearing in physics which can effectively be constructed and to represent them in a more direct, obvious way. Having defined the Poisson process (with the unit interval as the sample space, and the expected number of " signals " l in any interval of length l), they call any non-decreasing sequence (s_n) of non-negative numbers a signal sequence, and term such a sequence a Poisson sequence if the following is true: given any natural number k, any system of mutually disjoint left-open intervals $I_1, \ldots, I_k$ and non-negative integers $m_1, \ldots, m_k$, the " relative measure " $|B|_R$ of the set B of all $t \geqq 0$ such that each $t+I_r$ contains exactly m_r members of (s_n), $r = 1, \ldots, k$, is equal to

$$\prod_{r=1}^{k} \exp(-|I_r|)|I_r|^{m_r}/m_r!$$

Here, $|A|$ denotes the Lebesgue measure of a set A, and $|A|_R = \lim_{r \to +\infty} (1/T)|A \cap [0, T]|$. Substituting for B the set C of all natural numbers l such that $s_l + I_r$ contains exactly m_r members of (s_n), $r = 1, \ldots, k$, and for $|B|_R$ the " frequency " $\operatorname{fr} C = \lim_{N \to +\infty} (1/N)$ (cardinal number of $(C \cap \{1, 2, \ldots, N\})$), the definition is obtained of a " Poisson sequence in itself ". Neither of these concepts includes the other one, as is shown later by examples.

The authors recall that a sequence of measurable functions $(a_n(\tau))$, defined for $0 \leqq \tau \leqq 1$ and with values in $[0, 1[$, is regular if and only if

$$s_n(\tau) = -\log \prod_{r=1}^{n} (1 - a_r(\tau))$$

is a Poisson process. They then prove that

$$(*) \qquad s_n = -\log \prod_{r=1}^{n} (1 - a_r)$$

is a Poisson sequence in itself if and only if (a_n) is a

STEINHAUS, H. & URBANIK, K. (Math. Inst., Polish Academy of Sciences, Warsaw) **10.1** (2.5)
Poisson sequences—*In German*
Math. Zeit. (1960) **72**, 127-145 (9 references)
continued

regular sequence of numbers (perfectly uniformly distributed in Koroboff's sense), that is $0 \leqq a_n < 1$ and $\operatorname{fr}\left(\bigcap_{r=1}^{k} \{n : a_{n+r} < x_r\}\right) = \prod_{r=1}^{k} x_r$ for any system of numbers $x_1, \ldots, x_k$ in $[0, 1]$. If (a_n) is regular and

$$(**) \qquad \lim_{n \to +\infty} \left(\prod_{r=1}^{n} (1 - a_r)\right)^{1/n} = 1/e,$$

the sequence (s_n) defined by (*) is a Poisson sequence, but here neither condition is necessary, and (**) cannot be omitted. Effective examples of Poisson processes and " Poisson sequences in itself " are deduced from these theorems, and a sequence which is both a Poisson sequence and a " Poisson sequence in itself " is constructed.

It follows from the individual ergodic theorem that almost all sample sequences of a Poisson process are Poisson sequences and Poisson sequences in itself. The converse is true only under certain additional assumptions: a process is a Poisson process if it is " homogeneous " and almost all sample sequences are Poisson sequences, or if it is " stationary " and almost all sample sequences are Poisson sequences in itself. Finally, a set of sufficient conditions is given in order that a signal sequence (s_n) be a Poisson sequence in itself; using the author's terminology, (s_n) should be dispersed and measuring intervals, and $(s_{n+1} - s_n)$ should be an independent sequence.

(K. Krickeberg)

YUEH, M. I. (Institute of Mathematics, Academia Sinica)

On the problem of $M/M/n$ in the theory of queues—*In Chinese*
Acta. Math. Sinica (1959) **9**, 494-502

Let $M/M/n$ denote, as usual, a queueing process of the following type:

(i) Poissonian input of parameter λ;
(ii) queue-discipline: first come, first served;
(iii) service mechanism: n servers and negative-exponential service-time of parameter μ.

Let $p_k(t)$ be the probability that there are k " customers " at time t. In the case of single-server system $M/M/1$, the exact expression of $p_k(t)$ is already known. The author makes quotations from Khintchine & Feller: " In the Birth and Death process we have completely proved the existence of $p_k(= \lim_{t\to\infty} p_k(t))$. In the waiting line and servicing process such a proof will be possible, too. But the situation would be more complicated and it would be necessary to introduce essentially new ideas." . . . " We do not know the description of such a proof in any existing literature, though the problem itself was treated by many authors several times." . . . " From the explicit expression for $p_k(t)$, which we have not derived, however, it can be shown that the p_k thus obtained really represent the limiting distribution of the $p_k(t)$." Thus the author aims to get the explicit expression for $p_k(t)$, for general n, and to prove the existence of p_k for $n = 1, 2, 3$.

The author first establishes a simple algebraic expression for the sum of a sequence, the terms of which are defined by a recurrence formula. The Laplace transforms of the solutions for the system of differential equations of the process are proved to satisfy a difference equation, and to get the explicit expression for the solution of this difference equation the above-mentioned expression for the sum is used. The required explicit expression for $p_k(t)$ is given by the inversion formula of the Laplace transform. From this the existence of p_k for $n = 1, 2, 3$ follows.

(Z. Higuti)

COVEYOU, R. R. (Oak Ridge Nat. Lab., Tennessee) **11.5** (2.6)

Serial correlations in the generation of pseudo-random numbers—*In English*

J. Ass. Comp. Mach. (1960) **7**, 72-74 (7 references)

Many methods for the generations of pseudo-random numbers (distributed uniformly) can be expressed as follows: one chooses an integer P, the base; an integer λ, the multiplier, prime to P; an integer μ, the increment, less than P; and an integer a, the initial value. One then defines a sequence

$$x_0, x_1, x_2, \ldots$$

where

$$x_{n+1} \equiv \lambda x_n + \mu (\text{mod } P),$$

$$0 \leqq x_n < P \text{ and } x_0 = a.$$

The sequence will be periodic with period P or less. Work done on such sequences indicates that it is relatively easy to obtain sequences with sufficiently long periods. P is usually dictated by the computer to be 2^q for binary machines or 10^q for decimal machines where q is the word size.

It is the author's contention that λ and μ should be chosen to reduce the serial correlation in the sequence. If λ and μ are chosen so that the sequence has a sufficiently long period, then with error of order $1/P$ or less, the expected value of x_n is equal to $P/2$, the variance of x_n is equal to $P^2/12$, and the correlation coefficient between two successive numbers in the sequence is

$$\rho(x_n, x_{n+1}) = \frac{1 - 6\frac{\mu}{P}\left(1 - \frac{\mu}{P}\right)}{\lambda}.$$

The author states, " One must be cautious of the conclusions one draws from this formula because of the approximate nature of its derivation. One can say that a method which uses very small values of λ and μ is faulty." The method by which the correlation coefficient was computed can also be used to compute the serial correlation of higher order.

(D. Teichroew)

FREIBERGER, W. F. & JONES, R. H. (Brown University, Providence, R.I.) **11.5** (10.6)

Computation of the frequency function of a quadratic form in random normal variables—*In English*

J. Ass. Comp. Mach. (1960) **7**, 245-250 (10 references)

Let $x = (x_1, x_2, \ldots, x_n)$ be a sample of successive values taken from a discrete-valued stationary time series of normally distributed random variables with mean zero and covariance matrix $\boldsymbol{R}$. The spectral density function is estimated by a quadratic form Q. One of the outstanding problems in the theory of time series analysis is the problem of determining the distribution function of Q. If the number of observations is sufficiently large for the central limit theory to be applicable, normal approximations can be used to advantage. In recent years, work has aimed at the discovery of a more generally applicable approximate method and of more rational criteria for the sample size at which the large sample theory becomes useful; the state of the art has been summarised in two recent papers by Grenander, Pollak & Slepian [*J. Soc. Ind. Appl. Math.* (in press)], and Freiberger & Grenander [*Quart. Appl. Math.* (1959) **17**].

The authors review a number of attempts that have been made to compute the distribution of Q and then develop one which is an extension of a method of Gurland [*Ann. Math. Statist.* (1955) **26**, 122-127]. The density function of Q is computed by using the expression

$$g(x) = \frac{\lambda^{\alpha+1}}{\Gamma(\alpha+1)} x^{\alpha} e^{-\lambda x}[1 + c_3 L_3^{(\alpha)}(\lambda x) + c_4 L_4^{(\alpha)}(\lambda x) + \ldots$$

where

$$\alpha = \frac{m^2}{\sigma^2} - 1 \qquad \lambda = \frac{m}{\sigma^2}.$$

$L_i(\lambda x)$ is the i Laguerre polynomial, and the m and σ^2 are the mean and variance respectively. c_i are obtained from the cumulants which are expressed as functions of the eigenvalues of $\boldsymbol{RW}$ where $\boldsymbol{W}$ is the matrix of coefficients of Q. Some examples are given to indicate the computation time required on the IBM 650.

(D. Teichroew)

KEULS, M. (Landbouwhogeschool, Wageningen) **11.1** (9.4)

Tables and graphs of the power of the 5 and 10 per cent. *F*-tests for use in randomised block experiments—*In Dutch*

Statistica (1960) **14**, 111-134 (17 references, 3 tables, 3 figures 7 charts)

As randomised block experiments occur so frequently in agricultural and other research, it was thought worth while to adapt existing tables of the power of the 5 per cent. *F*-test to this special case (Table 1). A new table was prepared for the power of the 10 per cent. *F*-test (Table 3). The tables have been summarised in graphs ($\alpha = 0{\cdot}05$: figures 5 to 8; $\alpha = 0{\cdot}10$: figures 9 and 10) in order to facilitate their use and illustrations are given by five practical problems.

As an introduction, the author states that (in the terminology of Kuiper) the variance ratio is a random variable " isomorous " with the Fisher random variable $F_q^{p,\gamma}$. The following constants are used:

(i) n = number of treatments;
(ii) r = number of replications;
(iii) $p = n-1$ degrees of freedom as " treatments ";
(iv) $q = (r-1)(n-1)$ degrees of freedom as " error ";
(v) γ = " eccentricity " (non-centrality parameter $= A\sqrt{r}/\sigma$ where $A = \sqrt{\Sigma(\mu_i - \bar{\mu})}$ is the " discrepancy " of the expectations μ_i and r/σ^2 is the " accuracy " of the treatment means;
(vi) $\beta = \Pr(F_q^{p,\gamma} > c)$ is the power of the F_α-test if c satisfies $\Pr(F_q^p > c) = \alpha$ where α is the level of significance.

The power function relation determines a function $r \rightarrow \{A/\sigma\}\{r \mid \alpha, \beta, n\}$. Charts of this function are given for $\alpha = 0{\cdot}05$, $\beta = 0{\cdot}50$, $0{\cdot}70$, $0{\cdot}80$ and $0{\cdot}90$ (figures 5 and 6) and also for $\alpha = 0{\cdot}50$, $0{\cdot}10$; $\beta = 0{\cdot}50$, $0{\cdot}90$; $n = 2, 3, 5, 10$ and 20 (figures 7 and 9). The experimenter may choose his number r of replications so as to be reasonably sure (power of β_0) of rejecting H_0 when at least one among all differences $\mu_i - \mu_j$ attains a specified value of d. If $W = \max_{ij}(\mu_i - \mu_j)$ we notice whether $W/\sqrt{2} \leqq A \leqq \frac{1}{2}\sqrt{n}W$. Clearly, a satisfactory choice of r is obtained by taking the most unfavourable distribution of the expectations $\mu_i = A\sqrt{2} = W = d$.

A choice of r may then be obtained from

$$\{A/\sigma\}\{(r \mid \alpha, \beta, n)\} = \{d/\sigma\}\{\sqrt{2}\}:$$

σ is considered throughout the paper as well known from previous experimentation. This most unfavourable

KEULS, M. (Landbouwhogeschool, Wageningen) **11.1** (9.4) *continued*

Tables and graphs of the power of the 5 and 10 per cent. *F*-tests for use in randomised block experiments—*In Dutch*

Statistica (1960) **14**, 111-134 (17 references, 3 tables, 3 figures, 7 charts)

situation (which occurs when two μ_i deviate equally and in opposite direction from all the other $n-2\mu_i$ that do not differ) is rather unlikely for moderate values of n. The experimenter may also consider that in agricultural practice " a difference never comes alone ", that is to say a difference d, if present, will be accompanied by at least a few smaller differences that raise the power. Five simple models have been studied for the distribution of the μ_i and for each model the quotient A/W has been calculated for different values of n, (table 2). In model II exactly one μ_i differs from the others that are all equal; in model III the μ_i are taken as the averages in each of n equal classes of a normal distribution as given by table XX of Fisher and Yates. As a practical device, it is suggested that the most unfavourable of the models II and III should be taken, that is III for $n \leqq 12$, II for $n > 12$: a choice of r follows from charts 8 and 10 by

$$\{W/\sigma\}\{(r \mid \alpha, \beta, n)\} = d/\sigma$$

Remarks are made on the formulae used in computing the graphs.

(M. Keuls)

HAMBLEN, J. W. (University of Kentucky, Lexington) **11.5** (11.0)

Statistical programmes for the IBM 650—*In English*

Comm. of the Ass. Comp. Mach. (1959) **2**, 13-18 and 32-37

This paper, published in two parts, gives a brief description of a number of statistical programmes for the IBM 650 which are in use at various university computation centres.

The programmes are grouped into the following categories:

experimental designs;
correlation;
multiple regression;
factor analysis;
miscellaneous curve and surface fitting;
time series;
biserial correlation;
frequency distributions;
Chi Square, Phi coefficients, etc.;
random numbers;
and miscellaneous.

The paper states that copies of the programme decks can be obtained from the originating institution.

(D. Teichroew)

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ROTENBERG, A. (New York University) **11.5** (2.6)

A new pseudo-random number generator—*In English*

J. Ass. Comp. Mach. (1960) **7**, 75-77 (3 references, 3 tables)

In the most commonly-used method for generating pseudo-random numbers, one multiplication is required for each new random number. Since addition is usually faster than multiplication, attempts have been made to find a method by which random numbers could be generated by addition. Such methods have generally been unsatisfactory. The author has tested the sequence generated by

$$x_{i+1} = (2^a+1)x_i + c (\text{mod } 2^{35})$$

Four thousand and ninety-six numbers were generated and examined for distribution in eight equal intervals. The mean and variance of the distribution, as well as the runs up and down, and runs above and below the mean, are given. The results are regarded as being satisfactory.

The author proves that this sequence has the full period of 2^{35} for $a \geqq 2$ and c odd. The values selected were $a = 7$ and $c = 1$. The method saves 168 microseconds per random number generated on the IBM 704.

(D. Teichroew)

WYNN, P. (Mathematical Centre, Amsterdam) **11.5** (2.9)

The rational approximation of functions which are formally defined by a power series expansion—*In English*

Math. Comp. (1960) **14**, 147-186 (22 references, 13 tables)

" The advent of high speed digital computers and the consequent intensification of interest in the study of numerical analysis has caused considerable attention to be paid to the problem of obtaining approximation formulas for functions which occur in the theory of mathematical physics. It is the purpose of this note to describe the theory underlying various methods of obtaining rational approximations to functions which are formally defined by a power series expansion; the author assumes that the power series concerned are quite general in character, and that the functions with which they are associated do not satisfy a particular functional equation which would permit the use of any special method.

The theory is then subjected to a detailed analysis in terms of the computational steps involved, and a comparison, with regard to computational efficiency, of the various methods which may be devised for obtaining rational approximations is given."

The author uses the incomplete gamma function of large argument and the incomplete beta function as examples to illustrate his methods. He develops continued fraction expansions for these and other functions. The expansions are not new, but are presented from a computational point of view.

(D. Teichroew)